

# Symmetry, Self-Duality, and the Jordan Structure of Quantum Mechanics

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## Abstract

I explore several related routes to deriving the Jordan-algebraic structure of finite-dimensional quantum theory from more transparent operational and physical principles, mainly involving ideas about the symmetries of, and the correlations between, probabilistic models. The key tool is the Koecher-Vinberg Theorem, which identifies formally real Jordan algebras with finite-dimensional order-unit spaces having homogeneous, self-dual cones.

## 0 Introduction

These notes pull together some ideas for motivating the Jordan-algebraic structure of finite-dimensional quantum theory from principles having a more obvious operational or probabilistic meaning. The key tool is the Koecher-Vinberg theorem, which lets us identify formally real Jordan algebras with finite-dimensional order-unit spaces with homogeneous, self-dual cones. The strategy is to motivate homogeneity and self-duality of the cone of “effects” associated with a general probabilistic model, in terms of independently meaningful (and, ideally, plausible) principles.

Rather than offering a single set of axioms from which this structure can cleanly be derived, I explore in some detail the consequences of various assumptions, mainly to do with the symmetries of a system, and with the possibility of correlating this system with a canonical “conjugate” system. Afterwards, I observe that several different axiomatic packages can be extracted from the results of this study, any of which will enforce the homogeneity and self-duality of the cone generated by a system’s basic measurement outcomes.<sup>1</sup>

In a bit more detail, a finite dimensional *probabilistic model* specifies a set of basic measurements, a (compact) convex set of states — understood as probability weights on measurement outcomes — and a compact group of symmetries under which the both the set of measurements and the set of states are invariant. Any such model  $A$  gives rise, in a canonical manner, to an order-unit space  $\mathbf{E}(A)$ , in which the positive cone is generated by the model’s measurement outcomes. Any normalized, positive linear functional on  $\mathbf{E}(A)$

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<sup>1</sup>This is in accord with a prejudice of mine, namely, that quantum theory (at any rate, its probabilistic framework) *does not have* a single, stark physical meaning, but is more analogous to, say, the class of normal probability distributions, which arise in many different contexts *for many different reasons* — but which can be characterized in ways that lead us to expect this ubiquity. It also reflects the conviction that the various conditions considered here, and the structures that they constrain, are of independent interest, and merit a systematic study.

gives rise, by restriction, to a probability weight on measurement outcomes. I call the model *state-complete* if its state space contains every such weight.

Call a model *bi-symmetric* iff the group of symmetries acts transitively on pairs of distinct measurement outcomes, and on pure states. Where the state space is irreducible, bi-symmetry implies the existence of at most one  $G$ -invariant bilinear form on  $\mathbf{E}(A)$  that is positive on  $\mathbf{E}(A)_+$  and simultaneously orthogonalizes distinct measurement outcomes. Moreover, if it exists, this form is an inner product. If the model is also state-complete, it follows that the self-duality of the cone is equivalent to a condition called *sharpness*: every measurement outcome has probability one in a unique state.

It remains to secure the existence of an orthogonalizing invariant, positive form on  $\mathbf{E}(A)$ . I suggest three (related) ways of doing so. One is to postulate the existence, for every system  $A$ , of a *conjugate* system  $\overline{A}$ , canonically isomorphic to  $A$ , and a bipartite non-signaling state between  $A$  and its conjugate in which every measurement is perfectly, and uniformly, correlated with its image in  $\overline{A}$ . This state (analogous to the Bell state in quantum mechanics) then gives rise to the required bilinear form on  $\mathbf{E}(A)$ . Another approach is to require the existence of a bi-symmetric composite of two copies of  $A$ , and an invariant state in which the two component systems are independent. A third is to ask that all systems under consideration be representable as a set of objects in a dagger-monoidal category.

This work builds upon the earlier papers [8, 7, 6, 29, 30]. In particular, it echoes, but improves upon, the last of these. While I have included enough detail to make this paper reasonably self-contained, I do assume the reader has at least a glancing familiarity with the lingo of ordered vector spaces and convex cones, and more or less remembers what a Jordan algebra is. The book [13] by Faraut and Koranyi contains an excellent introduction to homogeneous self-dual cones and Jordan algebras, and includes a very accessible proof of the Koecher-Vinberg Theorem. See also [2] for a recent survey of this material with particular reference to quantum theory.

## 1 Order-Unit Spaces and Probabilistic Models

Let me begin by fixing some notation and terminology, recalling along the way some basic facts about ordered vector spaces. First, a convention: absent any statement to the contrary, *all vector spaces considered here are finite dimensional*. The dual space of a (finite-dimensional) vector space  $\mathbf{V}$  is denoted  $\mathbf{V}^*$ ; the space of linear transformations  $\mathbf{V} \rightarrow \mathbf{W}$  is denoted by  $\mathcal{L}(\mathbf{V}, \mathbf{W})$ , with  $\mathcal{L}(\mathbf{V})$  abbreviating  $\mathcal{L}(\mathbf{V}, \mathbf{V})$ .

By a *cone* in a real vector space  $\mathbf{V}$ , I will always mean a closed, convex, pointed, generating cone  $K$  — that is, a topologically closed convex set  $K \subseteq \mathbf{V}$ , closed under multiplication by non-negative scalars, satisfying  $K \cap -K = \{0\}$ , and spanning  $\mathbf{V}$  (whence,  $\mathbf{V} = K - K$ ). An

*ordered vector space* is a real vector space  $\mathbf{V}$  with a distinguished cone  $K =: \mathbf{E}_+$ . This determines a translation-invariant partial order, given by  $a \leq b$  iff  $b - a \in \mathbf{V}_+$ ; thus,  $\mathbf{V}_+ = \{a \in \mathbf{V} | a \geq 0\}$ . The *standard* or *pointwise cone* of  $\mathbf{V} = \mathbb{R}^X$  is the cone of non-negative functions. The standard cone in the space  $\mathcal{L}(\mathbf{H})$  of Hermitian operators on a (real or complex) Hilbert space  $\mathbf{H}$  consists of such operators of the form  $aa^*$ .

If  $\mathbf{V}$  and  $\mathbf{W}$  are two ordered vector spaces, a linear mapping  $\phi : \mathbf{V} \rightarrow \mathbf{W}$  is *positive* iff  $\phi(\mathbf{V}_+) \subseteq \mathbf{W}_+$ . Note that this is a cone. If  $\phi$  is a linear isomorphism and  $\phi^{-1}$  is positive — equivalently, if  $\phi(\mathbf{V}_+) = \mathbf{W}_+$  — then  $\phi$  is an *order-isomorphism* between  $\mathbf{V}$  and  $\mathbf{W}$ . An order-isomorphism  $\mathbf{V} \simeq \mathbf{V}$  is an order *automorphism* of  $\mathbf{V}$ . The *dual cone*  $\mathbf{V}_+^*$  is the set of positive linear functionals  $f \in \mathbf{V}^*$ .

An *order unit* on  $\mathbf{V}$  is a positive functional  $u \in \mathbf{V}_+^*$  that is *strictly positive*, i.e.  $u(x) = 0$  for  $x \in \mathbf{V}_+$  only if  $x = 0$ . This is equivalent (in finite dimensions, anyway), to the condition that, if  $f \in \mathbf{V}_+^*$ , then  $f \leq nu$  for some  $n \in \mathbb{N}$ . More generally, an order unit in an ordered vector space  $\mathbf{E}$  is an element  $u \in \mathbf{E}_+$  such that, for every  $a \in \mathbf{E}_+$ , there exists  $n \in \mathbb{N}$  with  $a \leq nu$ . An *order-unit space* is a pair  $(\mathbf{E}, u)$  where  $\mathbf{E}$  is an ordered vector space and  $u \in \mathbf{E}_+$  is an order unit. Order unit spaces arise very naturally (as we'll see below) as *probabilistic models*. One defines an *effect* to be an element  $a \in \mathbf{E}_+$  with  $a \leq u$ ; a discrete *observable* on  $\mathbf{E}$  is a set  $\{a_i\}$  of effects with  $\sum_i a_i = u$ . A *state* on  $\mathbf{E}$  is a positive functional  $\alpha \in \mathbf{E}_+^*$  with  $\alpha(u) = 1$ , so that, for any observable, the mapping  $a_i \mapsto \alpha(a_i)$  defines a probability weight on every observable  $\{a_i\}$ . We speak of  $\alpha(a_i)$  as the probability of  $a_i$  occurring when the observable  $\{a_i\}$  is measured in state  $\alpha$ .

As an illustration, if  $\mathbf{H}$  is a finite-dimensional complex Hilbert space, let  $\mathcal{L}(\mathbf{H})$  denote the space of Hermitian operators on  $\mathbf{H}$ , ordered by the cone of positive operators (operators of the form  $aa^*$ ). Then  $(\mathcal{L}(\mathbf{H}), \text{Tr})$  is an order-unit space, in which the observables are exactly the discrete “POVMs” representing quantum observables, and the states are the linear functionals  $a \mapsto \text{Tr}(\rho a)$  corresponding to density operators  $\rho$  on  $\mathbf{H}$ .

The order unit space  $\mathcal{L}(\mathbf{H})$  has two very striking geometric properties:

**Definition 1 (Self-Duality and Homogeneity):** An order-unit space  $\mathbf{E}$  is *self-dual* iff there exists an inner product  $\langle, \rangle$  on  $\mathbf{E}$  such that

$$\mathbf{E}_+ = \mathbf{E}^+ := \{a \in \mathbf{E} \mid \langle a, b \rangle \geq 0 \ \forall b \in \mathbf{E}_+\}.$$

An order-unit space  $\mathbf{E}$  is *homogeneous* iff the group of order-automorphisms (invertible positive mappings with positive inverses)  $\mathbf{E} \rightarrow \mathbf{E}$  acts transitively on the *interior* of  $\mathbf{E}_+$ .

Beyond  $\mathcal{L}(\mathbf{H})$ , the cone of (Jordan) squares in any formally real Jordan algebra is homogeneous and self-dual. Remarkably, this is the only example!

**Theorem (Koecher [18], Vinberg [25])** *Let  $\mathbf{E}$  be an homogeneous, self-dual (HSD) order-unit space. Then there exists a unique formally real Jordan product on  $\mathbf{E}$ , with respect to which  $\mathbf{E}_+$  is the cone of squares, and  $u$  is the identity.*

Both homogeneity and self-duality seem a bit more transparent to physical (or operational, or probabilistic) intuition, than does the Jordan product. So it's reasonable to try to motivate these two constraints independently. Homogeneity seems to present the easier challenge. Indeed, if we view order-automorphisms of  $\mathbf{E}$  as representing reversible physical processes on the corresponding system, then the homogeneity of the “state cone”  $\mathbf{E}_+^*$  simply requires that every interior *state* be reversibly transformable into any other by some physical process. Of course, the adjective “interior” is annoying here. In an earlier paper [7] with Howard Barnum and Philipp Gaebler, it is shown that homogeneity also follows from the assumption that every state on  $\mathbf{E}$  is the marginal of a bipartite “steering” state. This condition also makes the state cone *weakly* self-dual, that is, isomorphic to its dual cone. However, strict self-duality requires this isomorphism to be mediated by an inner product, and this has proved trickier to motivate.

A different approach, explored in [30], is to derive the homogeneity and self-duality of the “effect cone”  $\mathbf{E}_+$  from ideas about the symmetries of systems, and the possibility of correlating two copies of a system. In order to achieve this, I made use of an ad-hoc “minimization” axiom (which I’ll review below). Here, I aim to do better, and, in particular, to avoid this assumption.

### 1.1 Test spaces and probabilistic models

For my purposes, the abstract order-unit spaces dealt with above are a little *too* abstract.

**Definition 2 (Test spaces):** A *test space* is a pair  $(X, \mathfrak{A})$  where  $X$  is a set of *outcomes* and  $\mathfrak{A}$  is a covering of  $X$  by non-empty sets called *tests*, interpreted as the sets of mutually exclusive outcomes associated with various experiments. A *probability weight* on  $(X, \mathfrak{A})$  is a function  $\alpha : X \rightarrow [0, 1]$  with  $\sum_{x \in E} \alpha(x) = 1$  for every  $E \in \mathfrak{A}$ . I’ll write  $\Omega(X, \mathfrak{A})$  for the convex set of all probability weights on  $(X, \mathfrak{A})$ .<sup>2</sup>

The *rank* of a test space  $(X, \mathfrak{A})$  is the least upper bound of  $|E|$  where  $E \in \mathfrak{A}$ . For purposes of this note, *all test spaces have finite rank*. In particular, all tests are finite sets. It follows easily that the set  $\Omega(X, \mathfrak{A})$  of all probability weights on  $(X, \mathfrak{A})$  is a closed, and hence, compact, subset of  $[0, 1]^X$ .

*Notation:* Anticipating later results, I’ll write  $x \perp y$  to mean that outcomes  $x, y \in X$  are *distinguishable* by means of a test in  $\mathfrak{A}$  — that is, that  $x \neq y$  and there exists some  $E \in \mathfrak{A}$  with  $x, y \in E$ . Note that, at present, there is no linear structure in view, let alone an inner product, so the notation is only suggestive.

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<sup>2</sup>Mathematically, a test space is just a hypergraph. The terminology is meant to enforce a particular interpretation. Test spaces — originally termed “manuals” — were the basis for a generalized probability theory (and an associated “empirical logic”) developed in the 1970s and 80s by C. H. Randall and D. J. Foulis and their students. See [29] for a survey. It is important to understand that  $\mathfrak{A}$  is not necessarily intended as the complete catalogue of *all* possible measurements on a given system, but only some set of measurements sufficient to capture the system’s states, which we have singled out for some reason (perhaps one of tradition, or of exegetical efficiency).

**Definition 3 (Symmetry):** By a *symmetry* of a test space  $(X, \mathfrak{A})$ , I mean a bijection  $g : X \rightarrow X$  that permutes elements of  $\mathfrak{A}$ .

Notice that if  $g$  is a symmetry, then for all  $x, y \in X$ ,  $x \perp y$  iff  $gx \perp gy$ . An action of a group  $G$  on  $(X, \mathfrak{A})$  is an action by symmetries, and a test space equipped with such an action is a *G-test space*. I'll write  $\text{Aut}(X, \mathfrak{A})$  for the group of all symmetries of  $(X, \mathfrak{A})$ . In the cases that will interest us, this will always be isomorphic to a compact subgroup of  $GL(d)$  for a sufficiently large finite dimension  $d$ .

In constructing a model for a probabilistic system, we may want to privilege not only the “observables” represented by the tests  $E \in \mathfrak{A}$ , but also certain states and certain symmetries. This suggests the following

**Definition 4 (Probabilistic Models):** A *probabilistic model* — or, for purposes of this note, just a *model* — is a structure  $(X, \mathfrak{A}, \Omega, G)$ , where  $(X, \mathfrak{A})$  is a (finite-rank) test space,  $\Omega$  is a separating, pointwise-closed (hence, pointwise compact) convex set of probability weights on  $(X, \mathfrak{A})$ , and  $G$  is a compact group of symmetries of  $(X, \mathfrak{A})$  leaving  $\Omega$  invariant.

I'll call  $\Omega$  the *state space* of the model; probability weights  $\alpha \in \Omega$  are *states*. Where  $\Omega$  has finite affine dimension, I'll say that the model is finite-dimensional. *All models considered in this paper are finite-dimensional in this sense.* In the interest of sanity, I'll hereafter denote models by Roman capital letters  $A, B, \dots$ , writing (for instance)  $A = (X, \mathfrak{A}, \Omega, G)$ . It will often be convenient to label the components with the name of the model, as, e.g.,  $(X(A), \mathfrak{A}(A), \Omega(A), G(A))$ . I will use the terms “model” and “system” interchangeably.

It's time to look at some examples.

**Example 1: Classical models** Let  $E$  be a single, classical outcome-set (say, for a coin-flip, or rolling a die). Let  $X = E$ ,  $\mathfrak{A} = \{E\}$ , and  $G \leq S(E)$  be any group you like of permutations of  $E$ . Let  $\Gamma$  be any separating, permutation-invariant set of probability weights on  $E$ , and let  $\Omega$  be the closed convex hull of  $\Gamma$ . Alternatively, choose any separating closed convex set  $\Omega$  of probability weights, and let  $G$  be the group of permutations leaving  $\Omega$  invariant.

**Example 2: Quantum Models** Let  $\mathbf{H}$  be an  $n$ -dimensional complex Hilbert space. The corresponding *quantum model* is  $A(\mathbf{H}) := (X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(\mathbf{H}))$ , where

- $X(\mathbf{H})$  is the set of rank-one projection operators on  $\mathbf{H}$ ,
- $\mathfrak{A}(\mathbf{H})$  is the set of maximal pairwise orthogonal sets of such projections,
- $\Omega(\mathbf{H})$  is the set of states of the form  $x \mapsto \text{Tr}(\rho x)$ ,  $\rho$  a density operator on  $\mathbf{H}^3$  and
- $U(\mathbf{H})$  is the group of unitary operators on  $\mathbf{H}$ , acting on  $X$  by conjugation.

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<sup>3</sup>Gleason's Theorem tells us that  $\Omega(\mathbf{H}) = \Omega(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}))$  for  $\dim(\mathbf{H}) > 2$ ; for  $\dim(\mathbf{H}) = 2$ , the density matrices need to be put in by hand.

**Example 3: The Square Bit** For a much different, and much simpler, example, consider a test space  $(X, \mathfrak{A})$  consisting of two disjoint, two-outcome tests — say,  $X = \{a, a', b, b'\}$  and  $\mathfrak{A} = \{\{a, a'\}, \{b, b'\}\}$ . Then the space  $\Omega$  of all probability weights on  $(X, \mathfrak{A})$  is affinely isomorphic to the unit square in  $\mathbb{R}^2$ . The *square bit* is the model  $(X, \mathfrak{A}, \Omega, G)$  where  $G$  is the dihedral group acting on  $\Omega$  in the obvious way, and dually on  $(X, \mathfrak{A})$ .

**Example 4: Jordan Models** Let  $\mathbf{E}$  be a formally real Jordan algebra, and let  $X$  denote the set of primitive idempotents in  $\mathbf{E}$ . A *Jordan frame* is a pairwise orthogonal set of idempotents summing to the order unit. Letting  $\Omega$  denote the set of states on  $\mathbf{E}$  and  $G$ , the set of Jordan automorphisms of  $\mathbf{E}$ , we have a *Jordan model*  $(X, \mathfrak{A}, \Omega, G)$ .

## 1.2 Models Linearized

Let  $A$  be a model. Every outcome  $x \in X(A)$  determines an affine functional  $\hat{x} : \Omega(A) \rightarrow \mathbb{R}$  by evaluation:  $\hat{x}(\alpha) = \alpha(x)$ . Letting  $\text{Aff}(\Omega(A))$  denote the space of all real-valued affine functionals on  $\Omega(A)$ , we have then a natural — and, clearly,  $G$ -equivariant — mapping  $X(A) \rightarrow \text{Aff}(\Omega(A))$ . It is largely harmless to assume that this is injective, i.e., that  $\Omega(A)$  separates points of  $X$ . (If not, replace  $(X(A), \mathfrak{A}(A))$  by the obvious quotient structure.) From now on, I assume this is the case; that is, I make it a *standing assumption* that *all probabilistic models have separating sets of states*.

In view of this, it is convenient to identify  $x \in X(A)$  with the corresponding functional, so that  $X \subseteq \text{Aff}(\Omega(A))$ . I also assume, from this point on, that *all models are finite dimensional*, in the sense that  $\Omega(A)$  has finite affine dimension. It follows that  $\text{Aff}(\Omega)$  is a finite-dimensional real vector space. Let  $\mathbf{E} = \mathbf{E}(A)$  denote the span of  $X$  in  $\text{Aff}(\Omega)$ , ordered by the cone consisting of linear combinations of outcomes having non-negative coefficients:

$$\mathbf{E}_+ = \left\{ \sum_i t_i x_i \mid x_i \in X, t_i \geq 0 \right\}.$$

Note that this may be smaller than the cone  $\{a \in \mathbf{E} \mid a(\alpha) \geq 0 \forall \alpha \in \Omega\}$  inherited from  $\text{Aff}(\Omega)_+$ , and that, unlike the latter, it depends on the choice of  $X$ . Notice, too, that the action of  $G$  on  $X$  extends uniquely to a linear action on  $\mathbf{E}$ , given by  $(ga)(\alpha) = a(\alpha \circ g)$  for all  $a \in \mathbf{E}$  and all  $\alpha \in \Omega$ , and that  $\mathbf{E}_+$  is stable under this action. Finally, observe that, for every  $E \in \mathfrak{A}$ ,  $\sum_{x \in E} x = u$ , where  $u$  is the unit functional  $u(\alpha) \equiv 1$  for all  $\alpha \in \Omega$ . This last serves as an order-unit for  $\mathbf{E}$ .

I'll call the order-unit space  $(\mathbf{E}(A), u)$  the *linear hull* of the model  $A$ . Notice that every test  $E \in \mathfrak{A}(A)$  can now be regarded as a discrete observable on  $\mathbf{E}(A)$ . Thus, we can, to a large extent, regard a probabilistic model as an order-unit space equipped with a distinguished collection of observables (sufficient to separate points), invariant under a distinguished compact group of order-automorphisms, *and* with a distinguished convex set  $\Omega$  of states (of which, more in a moment).

**Examples:** In the case of a quantum model  $A = A(\mathbf{H})$ ,  $\mathbf{E}(A)$  can be identified with the order-unit space  $\mathcal{L}(\mathbf{H})$  of Hermitian operators on  $\mathbf{H}$ , ordered by the usual cone, with  $u$

the identity operator. In the case of the square bit,  $\mathbf{E}(A)$  is isomorphic to  $\mathbb{R}^3$ , equipped with a cone having a square cross-section. In the case of a Jordan model,  $\mathbf{E}(A)$  is canonically isomorphic, as an order-unit space, to the given Jordan algebra.

### 1.3 Sharpness and State-Completeness

If  $(\mathbf{E}, u)$  is any order-unit space, a *state* on  $\mathbf{E}$  is a positive linear functional  $\rho : \mathbf{E} \rightarrow \mathbb{R}$  that is normalized so that  $\rho(u) = 1$ . If  $A = (X, \mathfrak{A}, \Omega, G)$  is a model, with linear hull  $\mathbf{E}(A)$ , then any state  $\alpha \in \Omega$  defines a state on  $\mathbf{E}(A)$ , just by evaluation:  $\alpha(a) := a(\alpha)$ . Conversely, a state  $\rho$  on  $\mathbf{E}$  defines a state on the test space  $(X, \mathfrak{A})$  by restriction. In general, however, this *will not* lie in the designated state space  $\Omega(A)$  of the model.

Let  $\widehat{\Omega(A)}$  denote the set of all states on  $(X, \mathfrak{A})$  arising from states on  $\mathbf{E}(A)$ . Obviously,  $\Omega \subseteq \widehat{\Omega}$ . We may regard  $\widehat{\Omega}$  as the set of probability weights that are consistent with all of the linear relations among outcomes that are satisfied by the given state space  $\Omega(A)$ . Evidently, the assignment  $\Omega \mapsto \widehat{\Omega}$  is a closure on the poset of closed convex subsets of the (full) state space of  $(X, \mathfrak{A})$ . Let's agree to call a model *state-complete* iff  $\Omega = \widehat{\Omega}$ . In this case,  $\mathbf{E}_+$  coincides with the cone  $\mathbf{E} \cap \text{Aff}(\Omega(A))_+$ , i.e, every element of  $\mathbf{E}_+$  taking positive values on  $\Omega$  belongs to  $\mathbf{E}_+$ . (This last condition is called *saturation* in [8]).

**Example 5:** For an example of a non-state complete model, let  $E = \{x, y\}$  be a single two-outcome classical outcome-set, and consider the probability weights  $p_1, p_2$  given by  $p_1(x) = .6, p_1(y) = .4$ , and  $p_2(x) = .4, p_2(y) = .6$ . The set  $\{p_1, p_2\}$  is invariant under the obvious action of  $G = S_2 = S(E)$ , and separates  $x$  and  $y$ . Let  $A = (E, \{E\}, \Omega, S_2)$  where  $\Omega$  is the closed convex hull of  $p_1$  and  $p_2$ , i.e.,  $\Omega = \{tp_1 + (1-t)p_2 \mid 0 \leq t \leq 1\}$ . Then  $\mathbf{E}(A) \simeq \mathbb{R}^2$  with  $\mathbf{E}_+$  the first quadrant. The full state space  $\widehat{\Omega}$  consists of all probability weights on  $E$ , and is thus considerably larger than  $\Omega$ .<sup>4</sup>

All of the non-classical models discussed above are state-complete. State-completeness is a pretty reasonable condition to impose on a probabilistic model, at least in a finite-dimensional setting, and it will figure as a crucial hypothesis in many of the results below. Nevertheless, in order to keep clearly in view what does and what does not depend on it, I make *no standing assumption* of state-completeness. To help in keeping this in mind, I'll use the notation  $\mathbf{V}(A)$  for the space  $\mathbf{E}(A)^*$ , ordered not by the natural dual cone, but by the cone  $\mathbf{V}_+(A)$  generated by the designated state space  $\Omega(A)$ . State-completeness amounts to the condition that  $\mathbf{V}(A) = \mathbf{E}(A)^*$  (in which case, we also have  $\mathbf{E}(A) \simeq \mathbf{V}(A)^*$ ).

Another condition that will play a significant role in what follows is *sharpness*:

**Definition 5 (Sharpness):** A model  $A = (X, \mathfrak{A}, \Omega, G)$  is *sharp* iff, for every  $x \in X$ , there exists a unique state  $\delta_x \in \Omega$  with  $\delta_x(x) = 1$ .

In the earlier papers [29, 30], I called a model sharp iff, for every outcome  $x \in X(A)$ , there

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<sup>4</sup>Thanks to Jon Barrett for pointing out this sort of simple example.

exists a unique state  $\alpha \in \mathbf{E}^*(A)$  with  $\alpha(x) = 1$ . If  $A$  is state-complete (as was tacitly assumed in [28]), this coincides with the present notion. Sharpness (in one form or another) has a long history in the quantum-logical literature. In particular, it played a central role in Gunson's axiomatics for quantum theory [14]. A stronger form of sharpness, in which it is also required that each pure state render certain a unique outcome, is used by Hardy in [16].

## 1.4 Morphisms of Models

At several points I'm going to need to treat models categorically. There are various notions of morphism one might use, but the one that makes the most sense in the current context seems to be the following.

**Definition 6 (Morphisms):** A *morphism* from a model  $A$  to a model  $B$  is a pair  $(\phi, \psi)$ , where

- (i)  $\phi$  is a mapping  $X(A) \rightarrow X(B)$ , pushing tests of  $A$  forward to tests of  $B$ , and pulling states of  $B$  back to states on  $A$  — that is,

$$\phi(\mathfrak{A}(A)) \subseteq \mathfrak{A}(B) \text{ and } \phi^*(\Omega(B)) \subseteq \Omega(A)$$

(where  $\phi^*(\beta) = \beta \circ \phi$ .)

- (ii)  $\psi \in \text{Hom}(G(A), G(B))$ ;
- (iii)  $\phi(gx) = \psi(g)\phi(x)$  for all  $x \in X(A), g \in G(B)$ .

In practice, it will be convenient to regard  $\psi$  as defining an action of  $G(A)$  on  $X(B)$ , writing  $gy$  for  $\psi(g)y$  for  $g \in G(A)$  and  $y \in X(B)$ . When I wish to suppress explicit mention of  $\psi$  in this way, I'll simply write  $\phi$  for the pair  $(\phi, \psi)$ . From this point of view, (iii) says that  $\phi$  is equivariant. (Note, though, that the given action of  $G(A)$  on  $X(B)$  must be through elements of  $G(B)$ .)

An *isomorphism* of models is an invertible morphism; equivalently, a bijective mapping  $\phi : X(A) \rightarrow X(B)$ , equivariant with respect to an action of  $G(A)$  on  $X(B)$  (by members of  $G(B)$ ), taking  $\mathfrak{A}(A)$  bijectively onto  $\mathfrak{A}(B)$ , and inducing an affine isomorphism  $\phi^* : \Omega(B) \rightarrow \Omega(A)$ . In particular, every symmetry of a model  $A$  is a morphism from  $A$  to itself.

A morphism  $\phi : A \rightarrow B$  lifts naturally to a positive linear mapping between the corresponding linear hulls. To spell this out, notice that the affine mapping  $\phi^* : \Omega(B) \rightarrow \Omega(A)$  guaranteed by condition (i) of the definition, induces a linear map  $\phi^{**} : \text{Aff}(\Omega(A)) \rightarrow \text{Aff}(\Omega(B))$ , given by  $(\phi^{**}a)(\beta) = a(\phi^*(\beta)) = a(\beta \circ \phi)$ . Identifying  $x \in X(A)$  with the corresponding vector  $x \in \mathbf{E}(A) \leq \text{Aff}(\Omega(A))$ , and similarly taking  $\phi(x) \in \mathbf{E}(B) \leq \text{Aff}(\Omega(B))$ , it follows that  $\phi^{**}(x) = \phi(x)$ . Thus,  $\phi^{**}$  restricts to a linear mapping  $\phi : \mathbf{E}(A) \rightarrow \mathbf{E}(B)$  extending  $\phi : X(A) \rightarrow X(B)$ . Since this takes outcomes to outcomes, it sends  $\mathbf{E}_+(A)$  into



$\mathbf{E}_+(B)$ , that is,  $\phi$  is positive. Note, too, that if  $E \in \mathfrak{A}(A)$  and  $F = \phi(E) \in \mathfrak{A}(B)$ , we have  $\phi(u_A) = \sum_{x \in E} \phi(x) = \sum_{y \in F} y = u_B$ . Thus, we can regard  $A \mapsto \mathbf{E}(A)$  as the object part of a functor from probabilistic models and morphisms, to order-unit spaces and positive, unit-preserving linear maps. This observation will be put to use in due course.

## 2 Bi-Symmetric Models

I now wish to impose some constraints on the models under consideration. This section spells out some consequences of a package of symmetry assumptions which, taken together, assert that (i) all pure states, all outcomes, and all tests look the same, and (ii) individual tests have no (or little) internal structure, in the sense that the outcomes of any test can be permuted more or less freely by symmetries of the model, keeping the test fixed.

**Definition 7a (Full symmetry):** A test space  $(X, \mathfrak{A})$  is *fully symmetric* under the action of a group  $G$  iff (i) every test  $E \in \mathfrak{A}$  has the same cardinality, and (ii) every bijection  $f : E \rightarrow F$ ,  $E, F \in \mathfrak{A}$ , is implemented by some element  $g \in G$ , i.e.,  $gx = f(x)$  for every  $x \in E$ .

See [27, 28, 30] for more on this notion. Full symmetry entails that  $G$  act transitively on both  $\mathfrak{A}$  and  $X$ .

**Example 6:** Let  $\mathbf{E}$  be a formally real Jordan algebra, and let  $X$  denote the set of primitive (that is, atomic) idempotents in  $\mathbf{E}$ . Let  $\mathfrak{A}$  be the collection of all finite subsets of  $X$  summing to the unit element of  $\mathbf{E}$ , and let  $\Omega$  be the set of all  $\rho \in \mathbf{E}_+$  with  $\langle \rho, u \rangle = 1$  where  $\langle \cdot, \cdot \rangle$  is the canonical inner product on  $\mathbf{E}$ . Finally, let  $G$  be the group of all Jordan automorphisms of  $\mathbf{E}$ . Then  $(X, \mathfrak{A}, \Omega, G)$  is a probabilistic model. Moreover, it is fully symmetric ([13], Theorem IV.2.5).

A weaker condition than full symmetry, still sufficient for most of what follows, is that  $G$  act transitively on  $\mathfrak{A}$  and on the set of orthogonal (that is, distinguishable) *pairs* of outcomes:

**Definition 7b (2-Symmetry):**  $(X, \mathfrak{A})$  is *2-symmetric* under the action of  $G$  iff (i)  $G$  acts transitively on  $\mathfrak{A}$ , and (ii)  $G$  acts transitively on pairs of distinguishable measurement outcomes, that is, for all outcomes  $x, y, u, v \in X$  with  $x \perp y$  and  $u \perp v$ , then there exists some  $g \in G$  such that  $gx = u, gy = v$ . Note that any fully-symmetric test space is also 2-transitive.

In the context, not of a test space, but of a probabilistic model, I am also going to ask that  $G$  act transitively on the set of pure states. Thus,

**Definition 8 (Bi-symmetry):** A model  $(X, \mathfrak{A}, \Omega, G)$  is *fully bi-symmetric*, respectively *bi-symmetric*, iff (i)  $G$  acts fully symmetrically, resp., 2-symmetrically, on  $(X, \mathfrak{A})$ , and (ii)  $G$

acts transitively on extreme points of  $\Omega$ .

Bi-symmetric models can readily be constructed “by hand”, as follows [28]. Suppose  $E$  is a set, thought of as the outcome-set of a “standard test”, and suppose  $H$  is a group acting 2-transitively on  $E$ . Let  $G$  be any group with  $G \geq H$ , and let  $K \leq G$  be any subgroup of  $G$  with  $K \cap H = H_{x_o}$ , the stabilizer in  $H$  of some point  $x_o \in E$ . Set  $X = G/K$ , and embed  $E$  in  $X$  via  $hx_o \mapsto hK$ , where  $h \in H$ . (The condition that  $K \cap H = H_{x_o}$  guarantees that this is well-defined). Let  $\mathfrak{A}$  be the orbit of  $E$  in  $\mathcal{P}(X)$  under  $G$ , that is,  $\mathfrak{A} = \{gE | g \in G\}$ . Then  $G$  acts 2-symmetrically on  $(X, \mathfrak{A})$ . Now choose any  $\delta_o \in \Omega(X, \mathfrak{A})$ , and set  $\Omega$  be the closed convex hull of  $G\delta_o$ . See [27, 28] for more on this construction. The possibility of freely construcing bi-symmetric models in this way means, on the one hand, that bi-symmetry is a reasonably benign assumption, but also that it is not a very constraining one.

*Remark:* Individually, state-transitivity and state-completeness are very reasonable axioms: the former asks that we construct our state space in a natural way (as just outlined); the latter asks that we enlarge our state space, if necessary, in an equally natural way. However, there is a tension between these reasonable requirements, in that enlarging the state space to secure state-completeness may spoil state-transitivity. We can only rarely satisfy both conditions at once.<sup>5</sup>

## 2.1 SPIN forms

Until further notice,  $A = (X, \mathfrak{A}, \Omega, G)$  is a bi-symmetric model of rank  $n$ , and  $\mathbf{E} = \mathbf{E}(A)$ .

**Definition 9 (SPIN forms):** Let  $\mathcal{B} : \mathbf{E} \times \mathbf{E} \rightarrow \mathbb{R}$  be a bilinear form. I will say that  $\mathcal{B}$  is *positive* iff  $\mathcal{B}(a, b) \geq 0$  for all  $a, b \in \mathbf{E}_+$ , *normalized* iff  $\mathcal{B}(u, u) = 1$ , and *invariant* iff  $\mathcal{B}(ga, gb) = \mathcal{B}(a, b)$  for all  $g \in G$ . I’ll call a *symmetric* positive, invariant, normalized bilinear bilinar form on  $\mathbf{E}$  a *SPIN form* for short.<sup>6</sup>

There is a more or less canonical example, namely, the inner product

$$\langle a, b \rangle_G := \int_G a(g\delta_o) b(g\delta_o) dg$$

where  $\delta_o$  is any pure state (that is, extreme point) in  $\Omega$  and the integration is with respect to normalized Haar measure. Owing to the transitivity of  $G$  on the set of pure states, this is independent of the choice of  $\delta_o$

We can also define a *degenerate* SPIN form  $\mathcal{B}_o$ , defined by

$$\mathcal{B}_o(a, b) = \langle a, u \rangle_G \langle b, u \rangle_G$$

for all  $a, b \in \mathbf{E}$ . This turns out to be independent of the choice of the SPIN inner product (indeed, of the SPIN form) appearing on the right.

<sup>5</sup>This should not be too dismaying: a set of axioms *must* be in some tension with one another if they are to single out a narrow class of models.

<sup>6</sup>Of course, this is an absolutely dreadful choice of terminology; but I can’t seem to think of anything better at the moment. Suggestions?

Any SPIN form on  $\mathbf{E}$  is associated with two non-negative real constants:

- (1)  $r^2 := \mathcal{B}(x, x)$  for all  $x \in X$ , and
- (2)  $c := \mathcal{B}(x, y)$  for all  $x, y \in X$  with  $x \perp y$ .

Call these the *parameters* of  $\mathcal{B}$  (though, as we'll now see, they are not independent of one another).

**Lemma 1:** *Let  $\mathcal{B}$  be a symmetric, positive, invariant, normalized bilinear form on  $\mathbf{E}$ . Then the parameters  $r$  and  $c$  satisfy*

- (a)  $\mathcal{B}(x, u) = 1/n$  for all  $x \in X$ ;
- (b)  $r^2 + (n-1)c = 1/n$
- (c)  $r^2 \leq 1/n$ .
- (d) Let  $m$  and  $M$  denote, respectively, the minimum and maximum values of  $\mathcal{B}$  on  $X \times X$ . Then  $m \leq 1/n^2 \leq M$ .
- (e) If  $\mathcal{B}$  is positive-semidefinite,  $r^2 \geq 1/n^2 \geq c$ .
- (f) If  $\mathcal{B}$  is an inner product and  $r^2 = 1/n^2$ , then  $\mathbf{E}$  is one-dimensional.

*Proof:* For (a), note that  $\mathcal{B}(x, u)$  is a constant, again by transitivity of  $G$  on  $X$ , whence,  $n\mathcal{B}(x, u) = \sum_{x \in E} \mathcal{B}(x, u) = \mathcal{B}(u, u) = 1$ . For (b) note that if  $x \in E \in \mathfrak{A}$ , we have  $\mathcal{B}(x, u) = \sum_{y \in E} \mathcal{B}(x, y) = \mathcal{B}(x, x) + \sum_{y \in E \setminus \{x\}} \mathcal{B}(x, y) = r^2 + (n-1)c$ , which gives the desired inequality. Since  $c_{\mathcal{B}} \geq 0$ , this also yields (c), as  $r^2 \leq r^2 + (n-1)c_{\mathcal{B}}$ . For (d), note that for any  $E \in \mathfrak{A}$ , we have  $n^2m \leq \sum_{x, y \in E} \mathcal{B}(x, y) = \mathcal{B}(u, u) = 1$ ; similarly,  $n^2M \geq \mathcal{B}(u, u) = 1$ .

For (e), observe that if  $\mathcal{B}$  is positive semi-definite, then  $\|v\| := \sqrt{\mathcal{B}(v, v)}$  is a semi-norm, with  $\|x\| = r$  for every  $x \in X$ . Hence, by the triangle inequality, we have

$$1 = \|u\| = \left\| \sum_{x \in E} x \right\| \leq \sum_{x \in E} \|x\| = nr,$$

whence,  $r^2 \geq 1/n^2$ . It now follows from (b) that

$$(n-1)c = 1/n - r^2 \leq 1/n - 1/n^2 = \frac{n-1}{n^2},$$

giving us  $c \leq 1/n^2$ . Finally, for (f), suppose  $\mathcal{B}$  is an inner product. If  $r^2 = 1/n$ , then it follows from (b), as above, that  $(n-1)c = \frac{n-1}{n^2}$ , whence,  $c = 1/n^2$  as well. Hence, for any  $x \perp y$  in  $X$ , we have

$$\frac{1}{n^2} = c = \mathcal{B}(x, y) = \|x\| \|y\| \cos \theta = r^2 \cos \theta = \frac{1}{n} \cos \theta,$$

whence, the angle  $\theta$  between  $x$  and  $y$  is 0, i.e.,  $x = y$ . It follows that  $(X, \mathfrak{A})$  has rank  $n = 1$ , whence,  $\dim \mathbf{E} = 1$ .

Finally, if  $\mathcal{B}$  is positive semi-definite, then by (b) and (c), we also have

$$(n-1)c \leq 1/n - r^2 \leq 1/n - 1/n^2 = \frac{n-1}{n^2},$$

giving us (g).  $\square$

**Corollary 1:** *If  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are SPIN forms on  $\mathbf{E}$ , then for all  $a \in \mathbf{E}$ ,  $\mathcal{B}_1(a, u) = \mathcal{B}_2(a, u)$ .*

*Proof:* If  $x \in X$ , then  $\mathcal{B}_1(x, u) = \mathcal{B}_2(x, u) = 1/n$ , by Lemma 1. Since  $X$  spans  $\mathbf{E}$ , the result follows from the bilinearity of  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .  $\square$

A consequence is that the degenerate form  $\mathcal{B}_o(a, b) := \mathcal{B}(a, u)\mathcal{B}(b, u)$  is independent of the choice of  $\mathcal{B}$ . Henceforth, I refer to this SPIN form as the *uniform* SPIN form.

## 2.2 Minimizing and Orthogonalizing Forms

If  $\mathcal{B}$  is a SPIN bilinear form on  $\mathbf{E}$  and  $x \in X$ , we can define a probability weight  $\alpha_x$  on  $(X, \mathfrak{A})$  by  $\alpha_x(y) := n\mathcal{B}(x, y)$  for all  $y \in X$ . There is, however, no guarantee that this state will belong to the designated state space  $\Omega$ .

Since  $G$  is compact, and acts continuously on  $\mathbf{E}$ , its orbits are also compact. In particular,  $X$  is compact. It follows that every bilinear form — in particular, every SPIN bilinear form — achieves a maximum and a minimum value on  $X \times X$ .

**Definition 10 (Minimizing and Orthogonalizing SPIN forms):** A SPIN bilinear form  $\mathcal{B}$  on  $\mathbf{E}$  is *minimizing* iff  $\mathcal{B}(x, y)$  achieves its minimum value on  $X \times X$  at a point  $(x, y)$  with  $x \perp y$ .  $\mathcal{B}$  is *orthogonalizing* iff  $c_{\mathcal{B}} = 0$ , i.e.,  $\mathcal{B}(x, y) = 0$  for all  $x, y \in X$  with  $x \perp y$ .

Clearly, orthogonalizing implies minimizing. In the language of this paper, Proposition 1 of [30] asserts that *if  $A$  is bi-symmetric, sharp, state-complete model, and  $\mathbf{E}(A)$  admits a minimizing form, then  $\mathbf{E}(A)_+$  is self-dual*. It is also shown that, under these assumptions,  $\mathbf{E}(A)$  has an orthogonalizing form. One of the main goals of the present paper is to find sufficient conditions for such a minimizing form to exist.

The existence of an orthogonalizing form has many consequences. For one thing, if  $\mathcal{B}$  is orthogonalizing, then for every  $x \in X$ , the probability weight  $\delta_x := n\mathcal{B}(x, \cdot)$  assigns probability 1 to  $x$  and 0 to any outcome  $y \perp x$ .

If  $\mathcal{B}$  is a SPIN form on  $\mathbf{E}$ , let's agree to write  $\mathbf{E}^+$  for the set of vectors  $a \in \mathbf{E}$  such that  $\mathcal{B}(a, b) \geq 0$  for all  $b \in \mathbf{E}_+$  (even if  $\mathcal{B}$  is not an inner product). The positivity of  $\mathcal{B}$  guarantees that  $\mathbf{E}_+ \subseteq \mathbf{E}^+$ . If  $\mathbf{E}^+ = \mathbf{E}_+$ , I'll say that  $\mathbf{E}$  is *self-dual with respect to  $\mathcal{B}$* . The following is essentially proposition 1 from [30], but formulated more generally, for SPIN forms rather than SPIN inner products:

**Lemma 2:** *Let  $A$  be a sharp, state-complete, bi-symmetric model. If  $\mathcal{B}$  is a non-degenerate orthogonalizing SPIN form on  $\mathbf{E}(A)$ , then  $\mathbf{E}(A)$  is self-dual with respect to  $\mathcal{B}$ .*

*Proof:* We have  $\mathbf{E}_+ \subseteq \mathbf{E}^+$  in any case. Let  $x \in X$ . Since  $A$  is state-complete, the probability weight  $\delta_x := n\mathcal{B}(x, \cdot)$  belongs to  $\Omega$ . Since  $B$  is orthogonalizing,  $\delta_x(x) = 1$ . Since the model is sharp and state-complete,  $\delta_x$  is the unique state with this property, and hence, pure. Since the model is state-complete, bi-symmetry guarantees that any pure state  $\epsilon$  on  $\mathbf{E}$  has the form  $g\delta_x = n\mathcal{B}(g^{-1}x, \cdot)$  for some  $g \in G$ . Every extremal vector  $v \in \mathbf{E}^+$  with  $\mathcal{B}(v, u) = 1$  corresponds to a unique pure state  $\epsilon_v$  via  $\epsilon_v = \mathcal{B}(v, \cdot)$ . Since  $\mathcal{B}$  is non-degenerate, it follows that  $v = ng^{-1}x$  for some  $g \in G$ . But then  $v \in \mathbf{E}_+$ . It follows that  $\mathbf{E}^+ \subseteq \mathbf{E}_+$ .  $\square$

Thus, if there exists an orthogonalizing SPIN *inner product* on  $\mathbf{E}(A)$  ( $A$  sharp, state-complete, and bi-symmetric) then  $\mathbf{E}(A)$  is self-dual. In [30], the existence of a *minimizing* SPIN inner product was simply postulated. Most of the remainder of this paper is devoted to finding reasonable sufficient conditions for the existence of such an inner product.

### 2.3 Irreducible Systems

By the Corollary to Lemma 1, the ortho-complement  $u^\perp = \{x \in \mathbf{E} | B(x, u) = 0\}$  is independent of the SPIN form  $B$ . I'll say that the model  $A$  is *irreducible* in case  $u^\perp$  has no non-trivial  $G(A)$ -invariant subspace.<sup>7</sup> Things work especially nicely when  $A$  is irreducible in this sense.

**Lemma 3:** *Let  $A$  be irreducible, and suppose  $\mathcal{B}$  is any particular non-degenerate symmetric, invariant, normalized bilinear form on  $\mathbf{E}$ . Then all SPIN forms on  $\mathbf{E}$  have the form*

$$\mathcal{B}_\lambda(a, b) := \lambda\mathcal{B}(a, b) + (1 - \lambda)\mathcal{B}(a, u)\mathcal{B}(u, b)$$

for some real parameter  $\lambda$ .

By the remark following Corollary 1, regardless of the choice of  $\mathcal{B}$ , we have  $\mathcal{B}_0$  the uniform SPIN form, in conformity with our earlier usage.

*Proof:* Let  $\mathcal{B}'$  be any SPIN bilinear form. Since  $\mathcal{B}$  is non-degenerate, we have an operator  $\beta : \mathbf{E} \rightarrow \mathbf{E}$ , self-adjoint with respect to  $\mathcal{B}$ , such that  $\mathcal{B}'(a, b) = \mathcal{B}(\beta a, b)$  for all  $a, b \in \mathbf{E}$ . Since  $\mathcal{B}$  is  $G$ -invariant,  $\beta$  is  $G$ -equivariant, i.e.,  $\beta(ga) = g\beta(a)$  for all  $a \in \mathbf{E}$  and all  $g \in G$ .

Let  $u^\perp$  be the orthocomplement of  $u$  with respect to  $\langle, \rangle$ . By Corollary 1,

$$u^\perp = \{a \in \mathbf{E} | \mathcal{B}'(a, u) = 0\} = \{a \in \mathbf{E} | \mathcal{B}(a, u) = 0\},$$

so  $u^\perp$  is invariant under  $\beta$ . Let  $\beta_o$  denote the restriction of  $\beta$  to  $u^\perp$ , noting that this is still self-adjoint and  $G$ -equivariant. In particular,  $\beta_o$  has a real eigenvalue  $\lambda$ , and the eigenspace  $V_\lambda$  of  $\beta_o$  is an invariant subspace of  $u^\perp$ . Since the latter is irreducible, and

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<sup>7</sup>Admitting that this is again lousy terminology.

$V_\lambda \neq 0$  (by the non-degeneracy of  $\mathcal{B}$ ),  $V_\lambda = V$ .<sup>8</sup> Thus, for all  $a_o, b_o \in u^\perp$ , we have

$$\mathcal{B}'(a_o, b_o) = \lambda \mathcal{B}(a, b).$$

If  $a, b$  are now arbitrary vectors in  $\mathbf{E}$ , we can write  $a = a_o + a_1$  and  $b = b_o + b_1$ , where  $a_o, b_o \in u^\perp$  and  $a_1 = \mathcal{B}(a, u)u$ ,  $b_1 = \mathcal{B}(b, u)u$ . By Corollary 1, we have  $\mathcal{B}'(a_o, u) = \mathcal{B}(a_o, u) = 0$ , so that  $\mathcal{B}'(a_o, b_1) = \mathcal{B}(b, u)\mathcal{B}'(a_o, u) = 0$ ; likewise,  $\mathcal{B}'(a_1, b_o) = 0$ . Thus,

$$\mathcal{B}'(a, b) = \mathcal{B}'(a_o + a_1, b_o + b_1) = \mathcal{B}(a_o, b_o) + \mathcal{B}(a_1, b_1) = \lambda \mathcal{B}'(a_o, b_o) + \mathcal{B}(a, u)\mathcal{B}(b, u)$$

Since  $a_o = a - \mathcal{B}(a, u)u$  and  $b_o = b - \mathcal{B}(b, u)u$ , we can also write this as

$$\mathcal{B}'(a, b) = \lambda(\mathcal{B}(a, b) - 2\mathcal{B}(a, u)\mathcal{B}(b, u) + \mathcal{B}(a, u)\mathcal{B}(b, u))) + \mathcal{B}(a, u)\mathcal{B}(b, u).$$

Simplifying, this gives us  $\mathcal{B}'(a, b) = \lambda \mathcal{B}(a, b) + (1 - \lambda)\mathcal{B}(a, u)\mathcal{B}(b, u)$ , as promised.  $\square$

Suppose now that  $c_\lambda = \mathcal{B}_\lambda(x, y)$  for orthogonal  $x, y$ , and let  $c = \mathcal{B}(x, y)$  where  $\mathcal{B}$  is a chosen SPIN inner product (say, the standard one arising from group averaging). Then

$$c_\lambda = \lambda(c - 1/n^2) + 1/n^2 = \lambda \frac{n^2 c - 1}{n^2} + \frac{1}{n^2}.$$

which is 0 iff

$$\lambda = \frac{1}{n^2} \left( \frac{n^2}{1 - n^2 c} \right) = \frac{1}{1 - n^2 c}.$$

(Notice that  $1 - n^2 c = 0$  only for  $c = 1/n^2$ , which is to say, only if  $\mathcal{B} = \mathcal{B}_o$ , which we have ruled out by taking  $B$  to be an inner product, so this value of  $\lambda$  is legitimate.) It follows that there is *at most one* orthogonalizing SPIN form on  $\mathbf{E}$ , this corresponding to a non-negative value of  $\lambda$ . In order to guarantee that such a form exists, we need to know something more about the positivity of the forms  $\mathcal{B}_\lambda$ . As above, let  $\mathcal{B}$  be any chosen SPIN inner product on  $\mathbf{E}$ ; as in Lemma 1, let  $m$  denote the minimum value of  $\mathcal{B}(x, y)$  as  $x, y$  range over  $X$ , and recall that, by part (d) of Lemma 1, this never exceeds  $1/n^2$ .

**Lemma 4:** *Let  $A$  be irreducible, with  $\dim \mathbf{E}(A) > 1$ . Let  $\mathcal{B}$  denote any particular SPIN inner product on  $\mathbf{E}$  (say, the standard one arising from group averaging) on  $\mathbf{E}$ , and let  $m$  and  $M$  denote, respectively, the minimum and maximum values of  $\mathcal{B}$  on  $X \times X$ . Then, for any  $\lambda \in \mathbb{R}$ ,*

(a) *The minimum value  $m_\lambda$  of  $\mathcal{B}_\lambda$  is given by*

$$m_\lambda = \begin{cases} \lambda m + (1 - \lambda)/n^2 = \lambda(m - 1/n^2) + 1/n^2 & \text{if } \lambda \geq 0 \\ \lambda M + (1 - \lambda)/n^2 = \lambda(M - 1/n^2) + 1/n^2 & \text{if } \lambda < 0 \end{cases} \quad (1)$$

(b)  *$\mathcal{B}_\lambda$  is positive iff*

$$\frac{1}{1 - Mn^2} \leq \lambda \leq \frac{1}{1 - mn^2}$$

*where  $m$  and  $M$  are, respectively, the minimum and maximum values of  $\mathcal{B}$  on  $X \times X$ .*

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<sup>8</sup>This is simply the real form of Schur's Lemma.

(c)  $\mathcal{B}_\lambda$  is positive-semidefinite iff  $\lambda \geq 0$ , and an inner product iff  $\lambda > 0$ .

*Proof:* (a) If  $\lambda$  is non-negative, the minimum of  $\lambda\mathcal{B}(x, y) + (1 - \lambda)/n^2$  occurs where  $\mathcal{B}$  is minimized; if  $\lambda < 0$ , the minimum occurs where  $\mathcal{B}$  is maximized. (b) From part (a), we see that for  $\lambda \geq 0$ ,  $m_\lambda$  is non-negative iff  $\lambda(m - 1/n^2) \geq -1/n^2$ , or, equivalently, (recalling that  $m < 1/n^2$ , so that  $m - 1/n^2$  is negative)

$$\lambda \leq -\frac{1}{n^2(m - 1/n^2)} = \frac{1}{1 - mn^2}.$$

Similarly, if  $\lambda < 0$ ,  $m_\lambda \geq 0$  iff  $\lambda \geq 1/(1 - Mn^2)$ . (c) Now suppose that  $\mathcal{B}_\lambda(a, a) \leq 0$ . Then  $\lambda\|a\|^2 + (1 - \lambda)\mathcal{B}(a, u)^2 < 0$ . Since  $u$  is normalized, and  $\mathcal{B}$  is an inner product, we have  $\mathcal{B}(a, u)^2 \leq \|a\|^2$ . whence, as  $\mathcal{B}(a, u)^2 \geq 0$ , we must have  $\lambda \leq 0$ . Thus, if  $\lambda > 0$ ,  $\mathcal{B}_\lambda$  is positive-definite, that is, an inner product. Conversely, suppose  $\lambda \leq 0$ . Let  $r^2 = \mathcal{B}(x, x) \leq 1/n^2$  and  $r_\lambda^2 = \mathcal{B}_\lambda(x, x)$ , and recall that, since  $\mathcal{B}$  is an inner product, Lemma 1 (d) gives us  $r^2 > 1/n^2$  (the inequality strict, as  $\mathbf{E}$  is not one-dimensional.) Now we have  $r_\lambda^2 = \mathcal{B}_\lambda(x, x) = \lambda(r^2 - 1/n^2) + 1/n^2 < 1/n^2$ . But now, again by Lemma 1 (d),  $\mathcal{B}_\lambda$  is not an inner product.  $\square$

It now follows that, at the critical value  $\lambda = 1/(1 - cn^2)$  for which the form  $\mathcal{B}_\lambda$  is orthogonalizing,  $\mathcal{B}_\lambda$  is positive — a SPIN form — iff  $\lambda \leq 1/(1 - mn^2)$ , i.e.,

$$(1 - mn^2) \geq (1 - cn^2) \Leftrightarrow -mn^2 \leq -cn^2 \Leftrightarrow c \leq m$$

which occurs iff  $c = m$ , i.e., iff  $\mathcal{B}$  is minimizing. Thus, we have

**Corollary 2:** *Let  $A$  be irreducible. Then  $\mathbf{E}$  supports an orthogonalizing SPIN form iff it supports a minimizing SPIN form. In this case, the unique orthogonalizing SPIN form is the form  $\mathcal{B}_{\bar{\lambda}}$ , where  $\bar{\lambda} = \frac{1}{1 - mn^2}$ , the maximum value of  $\lambda$  for which  $\mathcal{B}_\lambda$  is positive. This is an inner product.*

We are free to replace the given SPIN inner product  $\mathcal{B}$  in Lemma 2 with any other SPIN inner product. Choosing the SPIN inner product  $\mathcal{B}_{\bar{\lambda}}$  for the maximal value  $\bar{\lambda}$ , we obtain a range of values  $0 \leq \lambda \leq 1$ . Henceforth, I assume this parametrization, so that  $\bar{\lambda} = 1$ . To emphasize that  $\mathcal{B}_1$  is an inner product, I'll sometimes write it as  $\langle, \rangle_1$ .

It follows that, where  $A$  is irreducible, the inner product  $\mathcal{B}_1 = \langle, \rangle_1$  is the *only candidate* for an orthogonalizing SPIN inner product. To put it another way: if  $A$  is irreducible, then there exists at most one orthogonalizing SPIN form on  $A$ , *and this is an inner product*.

### 3 Composites and Conjugates

Evidently, what is now wanted is a physically (or operationally, or probabilistically) natural condition guaranteeing the existence of an orthogonalizing (equivalently, minimizing) SPIN form. In this section I will offer three (not entirely independent) such conditions.

All turn on the notion of a composite system. Very briefly: there is a correspondence between *non-signaling* bipartite states and bilinear forms, so that equivariant bipartite states give rise to SPIN forms. The game is to seek conditions on such a state that (i) have a clear physical (or operational, or probabilistic) meaning, and (ii) guarantee that the corresponding SPIN state is orthogonalizing. I'll start with a quick review of how composite systems are handled in the current framework. More detail can be found in [4, 29].<sup>9</sup>

### 3.1 Composite systems and non-signaling states

**Definition 11 (Composites):** A *composite* of two models  $A$  and  $B$  is a model  $AB$ , plus an injection  $X(A) \times X(B) \rightarrow X(AB)$ , which I'll write as  $(x, y) \mapsto xy$ , such that

- (i) for all  $E \in \mathfrak{A}(A)$  and  $F \in \mathfrak{A}(B)$ ,  $EF := \{xy | x \in E, y \in F\} \in \mathfrak{A}(AB)$ ,
- (ii) for all  $\alpha \in \Omega(A)$ ,  $\beta \in \Omega(B)$ , there exists some  $\gamma \in \Omega(AB)$  with  $\gamma(xy) = \alpha(x)\beta(y)$ ; and
- (iii) for all  $g \in G(A)$ ,  $h \in G(B)$ , there exists some  $k \in G(AB)$  with  $k(xy) = (gx)(hy)$  for every  $x \in X(A)$ ,  $y \in X(B)$ .<sup>10</sup>

Condition (i) of the definition allows us to identify  $X(A) \times X(B)$  with the set  $X(A)X(B) = \{xy | x \in X, y \in Y\}$  of *product outcomes* in  $X(AB)$ . Let us write  $\mathfrak{A}(A) \times \mathfrak{A}(B)$  for the set of *product tests*, i.e., tests of the form  $EF$  provided for by condition (i). Evidently, every state in  $\Gamma$  restricts to a state on  $\mathfrak{A} \times$ ; by (ii), the set of such restrictions contains all *product states*  $\alpha \otimes \beta$ , defined by  $(\alpha \otimes \beta)(xy) = \alpha(x)\beta(y)$ . Also, by (iii), the stabilizer in  $G(AB)$  of the set  $X(A)X(B)$  extends the action on the latter of  $G(A) \times G(B)$ .

Another consequence of condition (i) is that

$$x_1 \perp x_2 \text{ in } X(A) \Rightarrow x_1y \perp x_2y \text{ in } X(AB)$$

for every choice of  $y \in X(B)$ ; likewise, if  $y_1 \perp y_2$  in  $X(B)$ , then  $xy_1 \perp xy_2$  for every  $x \in X(A)$ . This observation will be exploited below.

*Remark:* The category of *all* probabilistic models and morphisms has a natural product structure. Given models  $A$  and  $B$ , let  $A \times B$  be the model with outcome set  $X(A) \times X(B)$ , test space  $\mathfrak{A}(A) \times \mathfrak{A}(B) = \{E \times F | E \in \mathfrak{A}(A), F \in \mathfrak{A}(B)\}$ , state space the convex hull of the

<sup>9</sup>Another, unrelated, motivation is sketched in [30]. By choosing a fixed pure state  $\epsilon_o$ , we can represent elements of  $\mathbf{E}$  as continuous random variables on  $G$ , via  $x \in X \mapsto \hat{x} \in \mathbb{R}^G$ , where  $\hat{x}(g) = \alpha(gx)$ . That the canonical inner product obtained by group averaging be minimizing — which, in view of Corollary 2, is equivalent to the existence of an orthogonalizing form, at least for irreducible models, is equivalent to the condition that the covariance  $\text{cov}(\hat{x}, \hat{y})$  of two of these random variables be minimized precisely when the corresponding outcomes are distinguishable.

<sup>10</sup>The notation  $AB$  is not to be understood as referring (yet) to any particular *operation* of composition. That is,  $AB$  does not refer — as yet, anyway — to any *particular* composite.



product states, and symmetry group  $G(A) \times G(B)$ , acting as usual. (This is *not* a cartesian structure, since there are in general no morphisms  $A \times B \rightarrow A$  to serve as projections.) If we strengthen condition (ii) in Definition 11 to require that there exist a group homomorphism  $\psi : G(A) \times G(B) \rightarrow G(AB)$  with  $\psi(g, h)(xy) = (gx)(gy)$ , then  $g, h \mapsto k$  is a homomorphism, and  $x, y \mapsto xy$  defines a morphism  $A \times B \rightarrow AB$ .

A state  $\omega$  on a composite system  $AB$  is *non-signaling* [17] iff it has well-defined *marginal* (or reduced) states

$$\sum_{x \in E} \omega(xy) =: \omega_2(y) \text{ and } \sum_{y \in F} \omega(xy) =: \omega_1(x),$$

independent of the choice of tests  $E \in \mathfrak{A}(A)$ ,  $F \in \mathfrak{A}(B)$ . In this case, for every  $y \in X(B)$  and  $x \in X(A)$ , we define the *conditional states*  $\omega_{1|y}$  and  $\omega_{2|x}$  on  $(X, \mathfrak{A})$  and  $(Y, \mathfrak{A})$ , respectively, by

$$\omega_{1|y}(x) := \frac{\omega(xy)}{\omega_2(y)} \text{ and } \omega_{2|x}(y) := \frac{\omega(xy)}{\omega_1(x)}.$$

It is straightforward to establish the following bipartite *laws of total probability* for a non-signaling state  $\omega$ :

$$\omega_1 = \sum_{y \in F} \omega_2(y) \omega_{1|y} \text{ and } \omega_2 = \sum_{x \in E} \omega_1(x) \omega_{2|x} \quad (2)$$

for any choices of tests  $F \in \mathfrak{A}(B)$  and  $E \in \mathfrak{A}(A)$ .

**Definition 12:** A composite  $AB$  of models  $A$  and  $B$  is non-signaling iff all of its states are non-signaling, *and* all conditional states belong to the designated state spaces of  $A$  and  $B$  — that is,  $\omega_{2|x} \in \Omega(B)$  and  $\omega_{1|y} \in \Omega(A)$  for all  $x \in X(A)$  and  $y \in X(B)$ .

In particular, then, if  $AB$  is a non-signaling composite in the sense just defined, then  $\omega_1 \in \Omega(A)$  and  $\omega_2 \in \Omega(B)$  for every state  $\omega \in \Omega(AB)$ .

It is not hard to show (see [26]) that if  $\omega$  is non-signaling, then it gives rise to a unique bilinear form  $\mathcal{B}_\omega$  on  $\mathbf{E}(A) \times \mathbf{E}(B)$  with  $\mathcal{B}_\omega(x, y) = \omega(x, y)$  for all outcomes  $x \in X(A)$ ,  $y \in X(B)$ .<sup>11</sup> Thus, every non-signaling state  $\omega$  on  $AB$  is associated with a positive linear mapping  $\mathbf{E}(A) \rightarrow \mathbf{E}(B)^*$ , given by  $\hat{\omega}(a)(b) = \mathcal{B}_\omega(a)(b)$  for all  $a \in \mathbf{E}(A)$ ,  $b \in \mathbf{E}(B)$ . Since the conditional states  $\omega_{2|x}$  and  $\omega_{1|y}$  lie in  $\mathbf{V}(A)$  and  $\mathbf{V}(B)$ , respectively, the range of this mapping is contained in  $\mathbf{V}(B)$ , so we can — and I shall — regard  $\hat{\omega}$  as a positive linear mapping

$$\hat{\omega} : \mathbf{E}(A) \rightarrow \mathbf{V}(B).$$

If  $\langle, \rangle$  is a self-dualizing inner product on  $\mathbf{E}(B)$ , we can re-interpret  $\hat{\omega}$  as positive linear mapping  $\hat{\omega} : \mathbf{E}(A) \rightarrow \mathbf{E}(B)$ , given by the condition

$$\langle \hat{\omega}(a), b \rangle = \mathcal{B}_\omega(a, b).$$

---

<sup>11</sup>Conversely, if  $\omega$  is given by a bilinear form  $\mathcal{B}_\omega$ , it must be non-signaling, since we then have

$$\sum_{x \in E} \omega(x, y) = \sum_{x \in E} \mathcal{B}_\omega(x, y) = \mathcal{B}_\omega(u, y)$$

for all  $E \in \mathfrak{A}$ , and similarly in the second argument.

Notice that  $\hat{\omega}^*(b)(a) = \hat{\omega}(a)(b) = \omega(a, b)$ . Notice, too, that  $\hat{\omega}(u)(y) = \sum_{x \in E} \omega(x, y)$ , i.e.,  $\hat{\omega}(u)$  is the marginal of  $\omega$  in  $\Omega(X(B), \mathfrak{A}(B))$ , whence,  $\hat{\omega}(x)/u(\hat{\omega}(x))$  is just the conditional state  $\omega_{2|x}$ . Accordingly,  $\hat{\omega}$  is called the *conditioning map* associated with  $\hat{\omega}$ . Where this map is an order-isomorphism  $\mathbf{E}(A) \rightarrow \mathbf{V}(B)$ , we say that  $\omega$  is an *isomorphism state* [7].

*Jargon:* Let  $\omega$  be a non-signaling state on  $AA$ , and let  $\mathcal{B}_\omega$  be the corresponding bilinear form. If  $\mathcal{B}_\omega$  is a SPIN form, I'll call  $\omega$  a SPIN state.

A trivial but important example of a non-signaling state is the *uniform* (or *maximally mixed*) state on  $AA$ :  $\rho(z) = 1/n^2$  for every  $z \in X(AA)$ . The associated SPIN form, with parameters  $c = r^2 = 1/n^2$ , is exactly the degenerate, or uniform, SPIN form  $\mathcal{B}_0$ .

For later reference:

**Definition 13 (Local Tomography):** A composite  $AB$  is *locally tomographic* iff bipartite states in  $\Omega(AB)$  are uniquely determined by their values on product outcomes — that is, iff for all  $\omega_1, \omega_2 \in \Omega(AB)$ ,

$$\omega_1(x, y) = \omega_2(x, y) \forall x \in X(A), y \in X(B) \Rightarrow \omega_1 = \omega_2.$$

*Remark:* If  $AB$  is non-signaling, then (in our current, finite-dimensional setting), local tomography sets up a linear (NB: not ordered-linear) isomorphism  $\mathbf{E}(AB) \simeq \mathbf{E}(A) \otimes \mathbf{E}(B)$ . The cone on  $\mathbf{E}(A) \otimes \mathbf{E}(B)$  obtained by carrying forward the cone  $\mathbf{E}_+(AB)$  sits between the minimal (or projective) cone generated by the product states, and the maximal (or injective) cone consisting of all positive bilinear forms on  $\mathbf{E}^*(A) \otimes \mathbf{E}^*(B)$  [4, 26].

Both non-signaling and local tomography conditions are routinely assumed (sometimes explicitly, sometimes tacitly) in recent discussions of composite systems in generalized probabilistic theories ([15, 5, 23, 11], etc.). The non-signaling condition will be important in what follows, but the extremely powerful local tomography assumption plays no role here at all (but see further comments in the Conclusion).

### 3.2 Conjugate Systems

In view of the fact that equivariant non-signaling states yield SPIN forms, it is tempting simply to *postulate* the existence of a state  $\omega$  on a composite  $AA$  with the property that  $\omega(xy) = 0$  for all  $x \perp y$  in  $X(A)$ . Such a state would *perfectly correlate* every test  $E \in \mathfrak{A}$  with itself, in that, where Alice and Bob perform the same test at their locations, they are guaranteed the same outcome.

Unfortunately, in ordinary quantum theory, *there is no such state*: the candidate is the normalized trace, i.e.,  $B(x, y) = \text{Tr}(P_x P_y) = |\langle x, y \rangle|^2$ , which corresponds to no bipartite density operator. Fortunately, though, the strategy *does* work with a small modification. Consider a complex Hilbert space  $\mathbf{H}$  and its conjugate space  $\overline{\mathbf{H}}$ , and let  $\Psi \in \mathbf{H} \otimes \overline{\mathbf{H}}$  be the

(twisted?) Bell state

$$\Psi = \sum_{x \in E} x \otimes \bar{x}$$

where  $E$  is any orthonormal basis. This is independent of the chosen basis, and perfectly correlates every observable with its conjugate analogue — indeed,  $\langle \Psi, x \otimes \bar{y} \rangle = \langle x, y \rangle$ , so that  $|\langle \Psi, x \otimes y \rangle|^2 = |\langle x, y \rangle|^2$ .

This suggests the following idea. Recall from Section 1 that an *isomorphism* from a model  $A$  to a model  $B$  consists of a bijection  $\phi : X(A) \rightarrow X(B)$  taking  $\mathfrak{A}(A)$  bijectively onto  $\mathfrak{A}(B)$ , and such that  $\beta \mapsto \beta \circ \phi$  is an affine isomorphism from  $\Omega(B)$  to  $\Omega(A)$ , *plus* an action of  $G(A)$  on  $B$  by elements of  $G(B)$ , such that isomorphism  $\psi : G(A) \rightarrow G(B)$  such that  $\phi(gx) = g\phi(x)$  for all  $x \in X(A)$ ,  $g \in G(A)$ .

**Definition 14 (Conjugate Models):** A *conjugate* for a model  $A$  is a structure  $(\bar{A}, \gamma_A, \eta_A)$ , where  $\bar{A}$  is a model,  $\gamma_A : A \rightarrow \bar{A}$  is an isomorphism, and  $\eta_A$  is a bipartite state (on some non-signaling composite)  $A\bar{A}$  such that

$$\eta_A(x, \gamma_A(x)) = 1/n$$

for every  $x \in X(A)$ . I'll call  $\gamma_A$  the *conjugation map* and  $\eta_A$ , the *correlator* for the given conjugate.

**Example 7: Quantum Cases** If  $A = A(\mathbf{H})$  is a quantum model associated with a complex Hilbert space  $\mathbf{H}$ , let  $\bar{A} = A(\bar{\mathbf{H}})$ ; let  $\gamma_A : X(\mathbf{H}) \rightarrow X(\bar{\mathbf{H}})$  be the mapping  $x \mapsto \bar{x}$  (strictly speaking, the identity map!), and let  $\eta_A(x, \gamma_A(y)) = |\langle \Psi, x \otimes y \rangle|^2 = \text{Tr}(P_\Psi P_{x \otimes y})$ . As discussed above, this last is a correlator — obviously, symmetric and invariant.

**Lemma 5:** *If  $A$  has a conjugate, then it has a conjugate for which the correlator  $\eta_A$  is symmetric, in the sense that  $\eta(x, \gamma_A(y)) = \eta(y, \gamma_A(x))$ , and invariant, in the sense that  $\eta_A(gx, \bar{g}y) = \eta(x, \bar{y})$ .*

*Proof:* Let  $\eta^T(x, \gamma_A(y)) := \eta(y, \gamma_A(x))$ . Observe that this is again a correlator. Averaging the two gives us a symmetric correlator. Now suppose  $\eta$  is symmetric, and consider  $\eta^g(x, y) = \eta(gx, gy)$ . This again is a symmetric correlator, so averaging over the group yields an invariant symmetric correlator.  $\square$

Convention: Henceforth, assume that correlators are symmetric and invariant. It follows that

$$\mathcal{B}(a, b) := \eta(a, \gamma_A(b))$$

is an orthogonalizing SPIN form — and hence, if  $A$  is irreducible, an orthogonalizing SPIN inner product — on  $\mathbf{E}(A)$ .

**Theorem 1:** *Let  $A$  be bi-symmetric, and have a conjugate  $(\bar{A}, \gamma_A, \eta_A)$ . Then the following are equivalent:*

(a)  *$A$  is state-complete and  $\eta$  is an isomorphism-state.*

(b)  $A$  is self-dual with respect to the form  $\mathcal{B}(a, b) := \eta_A(a, \gamma(b))$ .

If  $A$  is irreducible, then  $B$  is an inner product, and (a) and (b) are equivalent to

(c)  $A$  is state-complete and sharp.

*Proof:* (a)  $\Rightarrow$  (b): If  $\eta$  is an order-isomorphism,  $\hat{\eta}^*$  takes  $\mathbf{E}_+$ 's extremal rays to those of  $\mathbf{V}_+$ . Since  $A$  is state-complete, the latter is  $\mathbf{E}_+^*$ . In particular,  $\hat{\eta}^*(\bar{x}) = \eta_2(\bar{x})\eta_{1|\bar{x}}$  is pure, and every pure state on  $\mathbf{E}(A)$  looks like this. We therefore have transitivity of  $G$  on pure states of  $A$ , and also that  $\hat{\eta}^*(\bar{x})(y) = \mathcal{B}(y, x) = \mathcal{B}(x, y)$ , so that  $\hat{\eta}_{1|\bar{x}} = n\mathcal{B}(x, \cdot)$  corresponds to a point in  $\mathbf{E}_+$ . Thus,  $\mathbf{E}^+ \subseteq \mathbf{E}_+$ .

(b)  $\Rightarrow$  (a) Conversely, suppose  $A$  is self-dual with respect to  $\mathcal{B}$ . Let  $\tau = \hat{\eta}^* \circ \gamma : \mathbf{E} \rightarrow \mathbf{V} = \mathbf{E}^*$  (the latter identity, one of linear spaces, not yet of ordered linear spaces). We have  $\tau(\mathbf{E}_+) \subseteq \mathbf{V}_+ \subseteq \mathbf{E}_+^*$ , so this is a positive mapping. Since  $\mathbf{E}$  is self-dual with respect to  $\mathcal{B}$ , we have  $\ker(\tau) \leq \mathbf{E}^+ = \mathbf{E}_+$ ; since the latter cone contains no subspaces other than 0,  $\tau$  is injective, and thus, in the present finite-dimensional setting, a linear isomorphism. The definition of  $\tau$  gives us  $\mathcal{B}(a, b) = \tau(b)(a)$  for all  $a, b \in \mathbf{E}(A)$ . Thus, for  $\beta = \tau(b)$ ,  $b \in \mathbf{E}$ , we have

$$\beta \in \mathbf{E}_+^* \Leftrightarrow \tau(b)(a) \geq 0 \ \forall a \in \mathbf{E}_+ \Rightarrow b \in \mathbf{E}_+$$

(the last, by self-duality), whence,  $\tau^{-1}(\mathbf{E}_+^*) \subseteq \mathbf{E}_+$ . In other words,  $\tau$  is an order-isomorphism. Since  $\gamma$  is also such, it follows that  $\hat{\eta}^*$  is an order-isomorphism, i.e.,  $\eta_A$  is an isomorphism state. Moreover, we have

$$\mathbf{E}_+^* = \tau(\mathbf{E}^+) = \tau(\mathbf{E}_+) \subseteq \mathbf{V}_+,$$

whence,  $\mathbf{E}_+^* = \mathbf{V}_+$ , i.e.,  $A$  is state-complete.

Suppose now that  $A$  is irreducible. Corollary 2 then tells us that the orthogonalizing SPIN form  $\mathcal{B}$  is an inner product. It follows from Lemma 2 that (c)  $\Rightarrow$  (a).

(b)  $\Rightarrow$  (c): We saw above that (b) implies state-completeness. Since  $A$  is irreducible, the orthogonalizing SPIN form  $\mathcal{B}$  is an inner product (indeed,  $\mathcal{B} = \mathcal{B}_1$ , in the notation of Section 2.3). We have  $\delta_x := n\langle x | \in \mathbf{V}^+$ , so that  $nx \in \mathbf{E}^+ = \mathbf{E}_+$ , with  $\langle nx, x \rangle = 1$ . Since  $x$  is extremal in  $\mathbf{E}_+$ ,  $\delta_x$  is extremal in  $\mathbf{V}_+ = \mathbf{E}_+^*$ . By state-transitivity, every pure state has the form  $\delta_y = n\langle y |$  for some  $y \in X$ . In particular, then, for every vector  $v \in \mathbf{E}^+$  with  $\langle v, x \rangle = \langle v, u \rangle = 1$ , we have  $\|v\| = \|nx\|$ . It follows that if  $\langle v, x \rangle = \langle x, x \rangle = 1$ ,  $v = x$ . Thus,  $A$  is sharp.  $\square$

Where a correlator  $\eta$  is an isomorphism state, I'll call it an *iso-correlator*. Using this jargon, we have

**Corollary 3:** *Let  $A$  be state-complete, bi-symmetric, irreducible, and have a conjugate with an iso-correlator. Then  $A$  is self-dual.*

In [29], I called a bipartite state  $\omega \in AB$  on two rank- $n$  test spaces *correlating* iff, for some pair of tests  $E \in \mathfrak{A}(A)$  and  $F \in \mathfrak{A}(B)$ , there exists a bijection  $f : E \rightarrow F$  such that

$\omega(x, f(y)) = 0$  for  $x \neq y$ . Evidently, the correlator of a conjugation is correlating in this sense (choose  $E \in \mathfrak{A}(A)$  and  $F = \gamma_A(E) \in \mathfrak{A}(\bar{A})$ , and let  $f(x) = \gamma_A(x)$  for  $x \in E$ ). The *correlation condition* of [29, 30] requires that every state on  $A$  arise as the marginal of some correlating bipartite state on a composite of two copies of  $A$ .

A stronger condition than the existence of a conjugate system, which will turn out to be useful, is the following

**Definition 15 (Strong Conjugates):** A *strong conjugate* for a model  $A$  consists of a system  $\bar{A}$ , an isomorphism  $\gamma_A : A \simeq \bar{A}$ , and a composite  $A\bar{A}$ , such that for every state  $\alpha \in \Omega(A)$ , there exists a non-signaling state  $\omega^\alpha \in \Omega(A\bar{A})$  satisfying

- (a)  $\omega_1^\alpha = \alpha$  (that is,  $\omega^\alpha$  is a dilation of  $\alpha$ )
- (b)  $\omega^\alpha(gx, g\bar{y}) = \omega(x, \bar{y})$  for all  $g$  fixing  $\alpha$ , and
- (c)  $\omega^\alpha$  is *correlating along  $\gamma_A$* , in the sense that there exists at least one test  $E \in \mathfrak{A}$  with  $\omega(x, \bar{x}) = \alpha(x)$  for all  $x \in E$  (where, as above,  $\bar{x} = \gamma_A(x)$ ).

Notice that a strong conjugate is (in effect) a conjugate, since we can take  $\eta$  to be  $\omega^\rho$ , where  $\rho$  is the uniform state on  $A$ .

**Example 8: The quantum case.** That the conjugate,  $A(\bar{\mathbf{H}})$ , of a quantum model  $A(\mathbf{H})$ , is in fact a strong conjugate is essentially just the Schmidt decomposition. Let  $\mathbf{H}$  be a Hilbert space, and, as above, let  $\bar{\mathbf{H}}$  denote the conjugate Hilbert space. For  $x, y \in \mathbf{H}$ , let  $x \odot y$  denote the operator on  $\mathbf{H}$  given by  $(x \odot y)z = \langle z, y \rangle x$ . In particular, if  $x$  is a unit vector, then  $x \odot x = P_x$ , the orthogonal projection operator associated with  $x$ . The mapping  $x, y \mapsto x \odot y$  is sesquilinear, that is, linear in its first, and conjugate linear in its second, argument. Hence, there is a natural linear isomorphism  $\mathbf{H} \otimes \bar{\mathbf{H}} \simeq \mathcal{B}(\mathbf{H})$  taking  $x \otimes \bar{y}$  to  $x \odot y$ . Suppose now that  $W$  is a density operator on  $\mathbf{H}$ , diagonalized by an orthonormal basis  $E \in \mathfrak{A}(\mathbf{H})$ . Then  $W$  has spectral resolution

$$W = \sum_{x \in E} \lambda_x P_x = \sum_{x \in E} \lambda_x x \odot x.$$

The corresponding vector in  $\mathbf{H} \otimes \bar{\mathbf{H}}$  is then

$$\Psi_W := \sum_{x \in E} \lambda_x x \odot \bar{x}.$$

If  $u, v \in E$  with  $u \perp v$ , then for every  $x \in E$ , we have either  $\langle x, u \rangle = 0$  or  $\langle u, y \rangle = 0$ , whence,

$$\langle \Psi_W, u \otimes \bar{v} \rangle = \sum_{x \in E} \lambda_x \langle x, u \rangle \langle \bar{x}, v \rangle = 0.$$

Moreover, on the diagonal, we have

$$\langle \Psi_W, u \otimes \bar{u} \rangle = \sum_{x \in E} \lambda_x |\langle u, x \rangle|^2 = \langle W u, u \rangle.$$

Evidently, the pure state corresponding to  $\Psi$  sets up a perfect correlation between  $E$  and its corresponding test  $\overline{E}$ , along the canonical isomorphism  $x \mapsto \overline{x}$ . Of equal note, if  $g$  is a unitary leaving  $W$  fixed, i.e, with  $gWg^{-1} = W$ , then the bipartite state (corresponding to)  $\Psi$  is also invariant under the diagonal action of  $G = U(\mathbf{H})$ :

$$\begin{aligned}\omega(gu, \overline{gv}) &= \sum_{x \in E} \lambda_x \langle x \otimes \overline{x}, gu \otimes \overline{gv} \rangle \\ &= \sum_{x \in E} \lambda_x \langle x, gu \rangle \langle \overline{x}, \overline{gv} \rangle \\ &= \sum_{x \in E} \lambda_x \langle g^{-1}x, u \rangle \langle \overline{g^{-1}x}, \overline{v} \rangle \\ &= \left\langle \left( \sum_{x \in E} \lambda_x g^{-1}x \otimes \overline{g^{-1}x} \right), u \otimes \overline{v} \right\rangle\end{aligned}$$

(where, for an operator  $a$  on  $\mathbf{H}$ ,  $\overline{a}$  denotes the linear operator on  $\overline{\mathbf{H}}$  given by  $\overline{a}(\overline{v}) = \overline{av}$  for all  $v \in \mathbf{H}$ .)

### 3.3 Factorizable States

Another way to motivate the existence of an orthogonalizing SPIN form on an irreducible system  $A$  is to suppose there exists an irreducible system  $B$  (perhaps another copy of  $A$ ) and a non-trivial SPIN form  $\mathcal{B}$  on a composite  $AB$  that *factors*, in the following sense:

**Lemma 6:** *Let  $AB$  be a bi-symmetric composite of bi-symmetric models  $A$  and  $B$ . Let  $\mathcal{B}$  be a SPIN form on  $\mathbf{E}(AB)$ , and suppose that  $\mathcal{B}$  factors, in the sense that, for all  $x, x' \in X(A)$  and all  $y, y' \in X(B)$ , we have  $\mathcal{B}(xy, x'y') = \mathcal{B}_1(x, x')\mathcal{B}_2(y, y')$  where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are normalized bilinear forms on  $\mathbf{E}(A)$  and  $\mathbf{E}(B)$ , respectively. Then  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are SPIN forms. If  $A$  and  $B$  are irreducible, then either (i)  $\mathcal{B}$  is uniform, or (ii)  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are orthogonalizing.*

*Proof:* That  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are both positive and symmetric is clear. To see that  $\mathcal{B}_1$  is invariant, note that

$$\mathcal{B}_1(a, b) = \mathcal{B}_1(a, b)\mathcal{B}_2(u_B, u_B) = \mathcal{B}(au_A, bu_B).$$

Thus if  $g \in G$ , we have

$$\begin{aligned}\mathcal{B}_1(ga, gb) &= \mathcal{B}_1(ga, \overline{gb})\mathcal{B}_2(u_B, \overline{u_B}) = \mathcal{B}(gau_B, \overline{gbu_B}) \\ &= \mathcal{B}((g, h)(au_B)(g, h)(b, u_B)) \\ &= \mathcal{B}(a, u_B, b, u_B) = \mathcal{B}_1(a, b)\end{aligned}$$

where  $h \in H$  is arbitrary; similarly for  $\mathcal{B}_2$ . Now let  $c, r$  be the parameters associated with  $\mathcal{B}$ , and let  $c_1, r_1$  and  $c_2, r_2$  be the parameters associated with  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , respectively. Let  $x \perp y$  in  $X(A)$  and  $z \perp w$  in  $X(B)$ . As observed above<sup>12</sup>, it follows that  $xz \perp yz$  and  $xz \perp xw$ , so we have

$$c = \mathcal{B}(xz, yz) = \mathcal{B}_1(x, y)\mathcal{B}_2(z, z) = c_1 r_2^2 \text{ and also } c = \mathcal{B}(xz, yw) = \mathcal{B}_1(x, y)\mathcal{B}_2(z, w) = c_1 c_2.$$

<sup>12</sup>See the remarks following Definition 11

Thus,  $c_1 r_1^2 = c_1 c_2$ . If  $B_1$  is not orthogonalizing, then  $r_1^2 = c_2$ . Since  $\mathcal{B}$  is irreducible,  $\mathcal{B}_2$  is uniform. But now the same reasoning, with the roles of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  reversed, tells us that  $c_1 c_2 = r_1^2 c_2$ , whence, as  $c_2 \neq 0$ , means that  $c_1 = r_1^2$ , whence,  $\mathcal{B}_1$  is also uniform. But then  $\mathcal{B}$  — hence,  $\omega$  — is uniform as well.  $\square$

**Definition 16:** Let  $A$  and  $B$  have conjugates  $\overline{A}$  and  $\overline{B}$ . A state  $\omega$  on  $(AB)(\overline{A}\overline{B})$  is *factorable* iff there exist states  $\omega_A$  on  $AA'$  and  $\omega_B$  on  $BB'$  such that  $\omega(xy\overline{x}\overline{y}) = \omega_A(x\overline{x})\omega_B(y\overline{y})$ .

Applying Lemma 6 to the bilinear forms associated with the non-signaling states  $\omega, \omega_A$  and  $\omega_B$ , we have

**Theorem 2:** Let  $A$  and  $B$  be irreducible, and let  $\omega$  be a factorable equivariant state on  $(AB)(\overline{A}\overline{B})$ . Then either  $\omega$  is the uniform state, or  $\omega_A$  and  $\omega_B$  are orthogonalizing.

Thus, if  $A$  is irreducible,  $B$  is a copy of  $A$ , and we can find a SPIN state on  $(AB)(\overline{A}\overline{B})$  making  $A\overline{A}$  and  $B\overline{B}$  independent, we are guaranteed an orthogonalizing SPIN form.

## 4 Monoidal Probabilistic Theories

In the categorical approach to quantum foundations [1, 2, 24], it is usually assumed — naturally enough — that a physical theory is a symmetric monoidal category  $\mathcal{C}$ , in which objects represent physical systems, morphisms represent physical processes, and the tensor product represents the physical composition of systems. A stronger, and perhaps more mysterious, assumption is that  $\mathcal{C}$  be *dagger*-monoidal, i.e, that it carry an involution compatible with the monoidal structure. In this section, I consider a symmetric monoidal category  $\mathcal{C}$  of bi-symmetric probabilistic models, and consider the associated “linearized” category  $\mathbf{E}(\mathcal{C})$  consisting of the linear hulls of the models in  $\mathcal{C}$ . The main result is that, if  $\mathbf{E}(\mathcal{C})$  is consistent with the existence of reasonable dagger-monoidal structure (the adjective “reasonable” being spelled out in Definitions 19 and 20 below), then there exists a factorable SPIN form on  $\mathbf{E}(A)$  for each model  $A \in \mathcal{C}$ ; hence, irreducible models in  $\mathcal{C}$  carry orthogonalizing SPIN inner products. Add the requirement that models in  $\mathcal{C}$  are sharp and state-complete, and all models in  $\mathcal{C}$  are self-dual.

### 4.1 Monoidal categories of probabilistic models

Henceforth,  $\mathcal{C}$  will denote a category of models, with morphisms as defined in section 1.4. It is reasonable to require — and I shall require — a bit more, namely, that every symmetry  $g \in G_A$  is in fact a morphism in  $\mathcal{C}$ , i.e.,  $G_A \leq \mathcal{C}(A, A)$ .

A *symmetric monoidal structure* on a category  $\mathcal{C}$  is a bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , plus a designated *unit object*  $I \in \mathcal{C}$ , and natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C, \quad \sigma_{A,B} : A \otimes B \simeq B \otimes A, \\ \lambda_A : I \otimes A \simeq A, \quad \text{and} \quad \rho_A : A \otimes I \simeq A$$

for all  $A, B, C \in \mathcal{C}$ . These isomorphisms are also required to satisfy various coherence conditions, e.g., that  $\lambda_A \circ \sigma_{A,I} = \rho_A$ . See [19] for details. A *symmetric monoidal category* is a category equipped with such a structure. A *dagger* on a SMC  $\mathcal{C}$  is an endo-functor  $\dagger : \mathcal{C} \rightarrow \mathcal{C}$  such that, for all objects  $A \in \mathcal{C}$ ,  $A^\dagger = A$ ,  $\sigma_{A,B}^\dagger = \sigma_{B,A}$ , and, for all morphisms  $\phi, \psi$  in  $\mathcal{C}$ ,  $\phi^{\dagger\dagger} = \phi$  and  $(\phi \otimes \psi)^\dagger = \phi^\dagger \otimes \psi^\dagger$ . A *dagger-monoidal category* is a SMC equipped with a dagger.

**Definition 18 (Monoidal probabilistic Theories):** A *monoidal probabilistic theory* is a symmetric monoidal category of probabilistic models, such that

- (i)  $G(A) \leq \mathcal{C}(A, A)$  for every  $A \in \mathcal{C}$ ,
- (ii) for every  $A, B \in \mathcal{C}$ , the monoidal product  $A \otimes B$  defines a non-signaling composite of  $A$  and  $B$ , in the sense of Definition 11
- (iii) the morphism  $A \times B \rightarrow A \otimes B$  sending  $x \in X(A), y \in X(B)$  to  $xy \in X(A \otimes B)$ , is a morphism in  $\mathcal{C}$ .

For each model  $A \in \mathcal{C}$ , we have the corresponding linear hull, the order-unit space  $\mathbf{E}(A)$ . Now, every morphism  $\phi \in \mathcal{C}(A, B)$  defines an affine mapping  $\phi^* : \Omega(B) \rightarrow \Omega(A)$ , given by  $\phi^*(\beta)(x) = \beta(\phi(x))$ . Pulling back again, we have a linear mapping  $\phi^{**} : \text{Aff}(\Omega(A)) \rightarrow \text{Aff}(\Omega(B))$ , which evidently takes  $\mathbf{E}(A)$  to  $\mathbf{E}(B)$ . Thus,  $A \mapsto \mathbf{E}(A)$  is the object part of a covariant functor from  $\mathcal{C}$  to the category of ordered linear spaces and positive linear mappings, taking  $\phi \in \mathcal{C}(A, B)$  to the corresponding positive linear mapping  $\phi : \mathbf{E}(A) \rightarrow \mathbf{E}(B)$  where  $\phi(a)(\beta) = a(\beta \circ \phi)$  for all  $a \in \mathbf{E}(A)$  and  $\beta \in \Omega(B)$ .

It is easy to check that the unit object for a monoidal probabilistic theory will necessarily be the trivial model  $T = (\{1\}, \{\{1\}\}, \{1\}, \{e\})$  having one outcome, one test, one state, and one symmetry. It is an annoying fact that there are *no* morphisms between  $T$  and any non-trivial model. The linearized category  $\mathbf{E}(\mathcal{C})$  will inherit the same defect. Thus, we'd like to extend the set of morphisms in the latter – at a minimum, we'd like to allow arbitrary linear mappings  $\mathbf{E}(T) \simeq \mathbb{R} \rightarrow \mathbf{E}(A)$  — representing elements of  $\mathbf{E}(A)$  — as well as some linear mappings  $\mathbf{E}(A) \rightarrow \mathbb{R}$ , representing states, to count as morphisms. This suggests the following

**Definition 19 (Representations):** A *representation* of a probabilistic theory  $\mathcal{C}$  is a functor  $\pi : \mathcal{C} \rightarrow \mathcal{E}$  where  $\mathcal{E}$  is a category of order-unit spaces, such that

- (i) for all  $V, W \in \mathcal{E}$ ,  $\mathcal{E}(V, W)$  is a space of linear mappings, ordered by a cone  $\mathcal{E}_+(V, W)$  of positive linear mappings;
- (ii)  $\pi(A) = \mathbf{E}(A)$  for every  $A \in \mathcal{C}$ , and  $\pi(\phi) = \phi$  for every  $\phi \in \mathcal{C}(A, B)$ ,
- (iii)  $\mathcal{E}(\mathbb{R}, \mathbf{E}(A)) \simeq \mathbf{E}(A)$ , for all  $A \in \mathcal{C}$ .



A representation is *self-dual*, resp. HSD, iff  $\pi(A)$  is self-dual, respectively HSD, for every model  $A \in \mathcal{C}$ .

If  $\mathcal{C}$  is a monoidal probabilistic theory, we can ask that  $\mathcal{E}$  also be symmetric-monoidal, and that  $\pi$  be a monoidal functor, i.e.,  $\pi(AB) = \pi(A) \otimes \pi(B)$  (at least up to a canonical isomorphism). In this case, we shall say that  $\pi$  is a *monoidal representation* of  $\mathcal{C}$ .

#### 4.1 Dagger-Monoidal Representations

A basic assumption in the categorical approach to finite-dimensional quantum theory [1, 2, 24] is that the category of physical systems and processes should be, not just a symmetric monoidal, but a *dagger*-symmetric monoidal category. Roughly, the monoidal product  $A, B \mapsto A \otimes B$  is understood to capture the idea of a composite of two non-interacting (but possibly entangled) systems; the meaning of the dagger is a perhaps a bit more mysterious, but is suggestive of an operation of time-reversal.

**Definition 20 (Dagger-monoidal representations):** A *dagger-monoidal representation* of a monoidal category  $\mathcal{C}$  of probabilistic models is a monoidal representation  $\pi : \mathcal{C} \rightarrow \mathcal{E}$  where

- (i)  $\mathcal{E}$  is  $\dagger$ -monoidal, with  $I \simeq \mathbb{R}$ ,
- (ii)  $u_A^\dagger \circ u_A = 1$  for all  $A \in \mathcal{C}$ , and
- (iii)  $\pi(g^{-1}) = \pi(g)^\dagger$  for all  $g \in G(A)$ ,  $A \in \mathcal{C}$ .

I'll say that  $\mathcal{C}$  is *dagger-monoidal* iff it has a dagger-monoidal representation.

Subject to assumptions (i) and (ii), there exists, for each  $A \in \mathcal{C}$ , a canonical  $G(A)$ -invariant, positive, symmetric bilinear form on  $\mathbf{E}(A)$  given by

$$\langle a, b \rangle := a \circ b^* \tag{3}$$

with  $a, b \in \mathbf{E}(A) \simeq \mathcal{E}(I, \mathbf{E}(A))$ .<sup>13</sup> Now, just by virtue of the monoidality of  $\mathcal{E}$ , this bilinear form *factors*, in the sense that

$$\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle.$$

This at once yields

**Theorem 3:** *Suppose  $\mathcal{C}$  is 2-symmetric, and admits a dagger-monoidal representation. Then the canonical form (3) is orthogonalizing on every irreducible system  $A \in \mathcal{C}$ .*

The proof is virtually identical to that of Theorem 2.

Combining this with Lemma 2 and Corollary 2, we have

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<sup>13</sup>In general, this is not an inner product; the angle-bracket notation is, however, standard in this context.

**Corollary 4:** Let  $\mathcal{C}$  be as in Theorem 3, and let  $A \in \mathcal{C}$  be state-complete, sharp and irreducible. Then the canonical bilinear form (3) is an inner product, with respect to which  $\mathbf{E}(A)$  is self-dual.

I'll call  $\mathcal{C}$  *†-Self Dual* (†-SD) iff there exists a †-monoidal representation  $\pi : \mathcal{C} \rightarrow \mathcal{E}$  where each  $\pi(A)$  is self-dual with respect to (3) — meaning, in particular, that this bilinear form is an inner product for every  $A \in \mathcal{C}$ . Corollary 2 tells us that if  $\mathcal{C}$  has a dagger-monoidal representation and every  $A \in \mathcal{C}$  is state-closed, sharp, and irreducible, then  $\mathcal{C}$  is dagger-SD.

*Remark:* Suppose that every object  $A \in \mathcal{C}$  has a conjugate  $(\bar{A}, \eta_A, \gamma_A)$  with  $\bar{\bar{A}} = A$ , in the sense that  $\bar{\bar{A}} = (A, \eta_A \circ \sigma, \gamma_A^{-1})$  (here  $\sigma : \bar{A} \times A \rightarrow A \times \bar{A}$  is the obvious swap mapping). If the correlators  $\eta_A$  are symmetric, in the sense that  $\eta_A(a, \bar{b}) = \eta_A(b, \bar{a})$  for all  $a, b \in \mathbf{E}(A)$ , then one can construct a dagger on  $\mathbf{E}(A)$  as follows: if  $\phi \in \mathcal{C}(A, B)$ , set

$$\phi^\dagger = \tau_A^{-1} \circ \phi^* \circ \tau_B$$

where  $\tau_A : \mathbf{E}(A) \rightarrow \mathbf{E}(A)^*$  is the mapping given by  $\tau_A(a) = \hat{\eta}^*(\gamma_A(a))$ , i.e.,  $\tau_A(a)(b) = \eta_A(b, \gamma_A(a))$ . With this definition of  $\phi^\dagger$ , one has  $\langle \phi(a), b \rangle = \langle a, \phi^\dagger(b) \rangle$  where  $\langle a, b \rangle := \eta_A(a, \gamma_A(b))$ , as above. (It would be interesting to know how the existence of a canonical, involutive operation  $A \mapsto \bar{A}$  of conjugation on a monoidal probabilistic theory comes to making the category dagger-compact.)

## 5 Image-closure

Thus far, our results mainly concern irreducible systems. One way of extending them to possibly reducible systems involves a condition I'll call *image-closure*.

**Definition 21 (Image of a model):** A morphism  $(\phi, \psi) : A \rightarrow B$  is *surjective* iff  $\phi(X) = Y$ ,  $\subseteq \phi(\mathfrak{A})$ ,  $H = \psi(G)$ , and  $\Omega(B) = \{\beta \in \Omega(X(B), \mathfrak{A}(B)) \mid \phi^*(\beta) \in \Omega\}$ . In this case, we call  $B$  the *image* of  $A$  under  $(\phi, \psi)$ , writing  $B = \phi(A)$ .

Notice that the image  $B = \phi(A)$  can be *simulated* by  $A$ , as follows. To prepare  $B$  in state  $\beta$ , prepare the  $A$  in the state  $\phi^*(\beta)$ . To measure  $F = \phi(E)$ , measure  $E$  on  $A$ , and, upon obtaining outcome  $x \in E$ , record  $\phi(x)$  as the outcome of  $F$ . To implement a symmetry  $h \in H$ , implement any corresponding symmetry  $g \in \psi^{-1}(h) \subseteq G$ . Operationally, it is reasonable (so long as we can prepare arbitrary states) to take  $\phi(A)$  as a legitimate physical model whenever  $A$  is.

**Definition 22 (Image-closure):** Call  $\mathcal{C}$  *image-closed* iff, for any model  $A \in \mathcal{C}$  and any surjective morphism  $\Phi = (\phi, \psi) : A \rightarrow B$ , (i) the model  $B$  belongs to  $\mathcal{C}$ , and (ii)  $\Phi \in \mathcal{C}(A, B)$ .

*Remark:* The image of a 2-symmetric model is 2-symmetric.

It would be rather embarrassing, at this point, if the category  $\mathcal{C}_{QM}$  of quantum models were not image-closed. In fact, however, a quantum model has no non-trivial images at all.

**Definition 23 (Incompressible Models):** A model  $A$  is *incompressible* iff, for all models  $B$ , any surjective homomorphism  $\phi : A \rightarrow B$  is either an isomorphism, or is trivial in the sense that  $X_B$  is a single point.

**Lemma 7:** *Every quantum model is incompressible.*

*Proof:* see this, notice first that if  $\phi : A \rightarrow B$  is a surjective morphism of models, with  $G$  acting transitively on  $X(A)$ , then  $X(B)$  is a transitive  $G$ -set. Hence,  $X(B) \simeq G(A)/G(A)_y$ , where  $G(A)_y$  is the stabilizer of any  $y \in X(B)$  under the action of  $G(A)$  on  $X(B)$ . Since  $\phi$  is equivariant,  $G(A)_x \leq G(A)_y$  for any  $x \in X(A)$  with  $\phi(x) = y$ . Now suppose that  $A$  is quantum, i.e.,  $A = A(\mathbf{H})$  for a complex  $n$ -dimensional Hilbert space  $\mathbf{H}$ . Then  $X(A) = X(\mathbf{H})$ , the set of rank-one projections on  $\mathbf{H}$ , and  $G(A) = U(\mathbf{H}) \simeq U(n)$ , with  $G(A)_x \simeq U(n-1)$ . Now,  $U(n-1)$  is a maximal proper subgroup of  $U(n)$ <sup>14</sup>. Hence, as  $G(A)_x \leq G(A)_y$ , we have either  $G(A)_y = G(A)_x$  or  $G(A)_y = G(A)$ . In the former case,  $\phi$  is a bijection, in the latter,  $X(B)$  is a point. Thus, a non-trivial surjective image of a quantum model  $A(\mathbf{H}) = (X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(n))$  has the form  $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Gamma, U(n))$ , where  $\Gamma$  is the set of all states on  $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}))$  of the form  $\rho \circ \phi$  where  $\rho$  is (the state associated with) a density operator on  $\mathbf{H}$  and  $\phi$  is a symmetry of  $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}))$ . By Wigner's Theorem,  $\phi$  has the form  $\phi(x) = V^{-1}xV$  where  $V : \mathbf{H} \rightarrow \mathbf{H}$  is either unitary or anti-unitary. We have, for every unit vector  $x \in X(\mathbf{H})$ ,

$$(\rho \circ \phi)(x) = \text{Tr}(\rho \phi(x)) = \text{Tr}(\rho V^{-1}xV) = \text{Tr}(V\rho V^{-1}x)$$

That is, the state  $\rho \circ \phi$  is the state on  $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}))$  associated with the linear operator  $V \circ \rho \circ V^{-1}$  — which is a linear, and thus a density operator, regardless of whether  $V$  is linear or anti-unitary. Thus,  $\Gamma = \Omega(\mathbf{H})$ .<sup>15</sup>  $\square$

*Remark:* This argument shows that any model  $A$  such that (i)  $X(A)$  transitive under  $G(A)$ , (ii) the stabilizer  $G(A)_x$  of an outcome  $x \in X(A)$  is a maximal subgroup of  $G(A)$ , and (iii)  $\Omega(A)$  invariant under symmetries of  $(X(A), \mathfrak{A}(A))$ , is incompressible.

Our goal now is to prove the following:

**Theorem 4:** *Suppose  $\mathcal{C}$  is an image-closed category of bi-symmetric models, having a dagger-monoidal representation. Then  $\mathbf{E}(A)$  carries an orthogonalizing SPIN inner product for  $A \in \mathcal{C}$ .*

Let  $(X, \mathfrak{A}, \Omega, G)$  be a fully symmetric model. Let  $\langle, \rangle$  be a SPIN inner product on  $\mathbf{E}$ , e.g., the one arising from group averaging. Suppose  $\mathbf{M} \leq u^\perp$  is a  $G$ -invariant subspace of  $u^\perp$ . Let  $P : \mathbf{E} \rightarrow \mathbf{M}$  be the corresponding projection operator (defined w.r.t. the standard inner

<sup>14</sup>Thanks to David Feldman for pointing this out

<sup>15</sup>If  $\dim(\mathbf{H}) > 2$ , Gleason's Theorem gives us  $\rho \circ \phi \in \Omega$  even more trivially. The preceding argument also works if  $\dim(\mathbf{H}) = 2$ .

product). For every  $x \in X$ , set

$$x_1 := P(x) + u/n$$

Then  $\sum_{x \in E} x_1 = P(u) + u = u$  (with  $P(u) = 0$  since  $\mathbf{M} \leq u^\perp$ ). Let  $X_1 = \{x_1 | x \in X\}$ ; for each  $E \in \mathfrak{A}$ , set  $E_1 = \{x_1 | x \in E\}$ , and let  $\mathfrak{A}_1 = \{E_1 | E \in \mathfrak{A}\}$ . Then  $(X_1, \mathfrak{A}_1)$  is a fully symmetric  $G$ -test space. Since  $X$  spans  $\mathbf{E}$ ,  $X_1$  spans  $\mathbf{E}_1 := \mathbf{M} \oplus \langle u \rangle$ . Let  $\mathbf{E}_{1+}$  denote the cone in  $\mathbf{E}_1$  consisting of non-negative linear combinations of elements of  $X_1$ .

**Lemma 8:** *There exists a  $G$ -invariant, separating set  $\Omega_1$  of states on  $(X_1, \mathfrak{A}_1)$  such that (i)  $A_1 := (X_1, \mathfrak{A}_1, \Omega_1, G)$  is a bi-symmetric model (in particular,  $G$  acts transitively on the extreme points of  $\Omega_1$ ), and (ii) the pair  $(\phi, id_G)$ , with  $\phi : X \rightarrow X_1$  given by  $\phi(x) = x_1$ , is a morphism of models.*

*Proof:* Let  $v \in \mathbf{E}^+$  represent a pure state, i.e., an extreme point of  $\Omega$ . Set  $v_1 = P(v) + u$ , and note that

$$\langle v_1, u \rangle = \langle P(v), u \rangle + \langle u, u \rangle = 1$$

and, for all  $x \in X$ ,

$$\langle v_1, x_1 \rangle = \langle P(v) + u, P(x) + u/n \rangle = \langle P(v), P(x) \rangle + 1/n = \langle v_1, x \rangle.$$

There is no guarantee that this last will be positive for all  $x \in X$ ; however, we can choose  $\epsilon > 0$  so that

$$v_\epsilon := \epsilon v_1 + (1 - \epsilon)u$$

belongs to  $\mathbf{E}^+$  – and hence, to  $\mathbf{E}_1^+$  – since  $u$  lies in the interior of  $\mathbf{E}^+$ . Now let

$$\Omega_1 := \overline{\text{co}}(Gv_\epsilon)$$

This is clearly a closed, convex,  $G$ -invariant set of states on  $(X_1, \mathfrak{A}_1)$ . We must show it is separating. Suppose

$$\langle gv_\epsilon, x_1 \rangle = \langle gv_\epsilon, y_p \rangle \tag{4}$$

for all  $g \in G$ . We have

$$\begin{aligned} \langle gv_\epsilon, x_1 \rangle &= \langle \epsilon v_1 + (1 - \epsilon)u, g^{-1}x_1 \rangle \\ &= \langle \epsilon v_1 + (1 - \epsilon)u, g^{-1}Px + u/n \rangle \\ &= \epsilon (\langle v_1, g^{-1}Px \rangle + \langle v_1, u/n \rangle) + (1 - \epsilon) (\langle u, g^{-1}Px \rangle + \langle u, u/n \rangle) \\ &= \epsilon \langle v_1, g^{-1}Px \rangle + \epsilon 1/n + (1 - \epsilon) 1/n \\ &= \epsilon \langle v_1, g^{-1}Px \rangle + 1/n. \end{aligned}$$

Similarly,  $\langle v_\epsilon, y_1 \rangle = \epsilon \langle v_1, g^{-1}Py \rangle + 1/n$ . Thus, (4) implies  $\langle v_1, g^{-1}Px \rangle = \langle v_1, g^{-1}Py \rangle$ , whence,  $\langle P(v) + u, g^{-1}P(x) \rangle = \langle P(v) + u, g^{-1}P(y) \rangle$ , whence,  $\langle P(v), g^{-1}P(x) \rangle = \langle P(v), g^{-1}P(y) \rangle$ . But this last is

$$\langle gv, Px \rangle = \langle gv, Py \rangle$$

for all  $g \in G$ . Since  $\{gv | g \in G\}$  is the full set of extreme points of  $\Omega$ , it is separating for  $\mathbf{E}$ . It follows that  $P(x) = P(y)$ , i.e.,  $x_1 = y_1$ .  $\square$

If  $\mathbf{M}$  is a minimal proper  $G$ -invariant subspace of  $u^\perp$ , the model  $A_1$  is irreducible. Hence, if  $A_1$  supports an orthogonalizing positive symmetric invariant bilinear form, then (by Corollary 2) this form is the standard SPIN inner product  $\langle \cdot, \cdot \rangle_1$  on  $\mathbf{E}_1$ .

*Remark:* The foregoing proof shows that if  $A$  is incompressible, then  $u^\perp$  is irreducible in  $\mathbf{E}(A)$ . Thus, we can dispense with image-closure, if we are willing to focuss our attention on incompressible models:

**Theorem 4b:** *Let  $\mathcal{C}$  be a dagger-monoidal category of 2-symmetric probabilistic models. Then for every incompressible model  $A \in \mathcal{C}$ ,  $\mathbf{E}(A)$  hosts an orthogonalizing SPIN inner product. If  $A$  is also state-closed and sharp, then  $\mathbf{E}(A)_+$  is self-dual.*

Returning now to a general situation, let  $u^\perp = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_k$  with each  $\mathbf{M}_j$  an irreducible invariant subspace for  $G$ . Let  $P_1, \dots, P_k$  be the corresponding projections, and, for each  $x \in X$ , let  $x_j = P_j(x) + u/n$ ,  $j = 1, \dots, k$ . Lemma 8 gives us, for each  $j$ , a bi-symmetric model  $(X_j, \mathfrak{A}_j, \Omega_j, G)$ , and, with this, a space  $\mathbf{E}_j = \mathbf{M}_j \oplus \langle u \rangle$  (ordered by the cone spanned by  $X_j$ ). Finally, since each  $A_j$  is irreducible, Corollary 2 gives us a standard (maximal) SPIN inner product  $\langle \cdot, \cdot \rangle_j$  on  $\mathbf{E}_j$ .

**Lemma 9:** *If  $\langle \cdot, \cdot \rangle_j$  is orthogonalizing for each  $j$ , then there exists an orthogonalizing inner product on  $\mathbf{E}$ .*

*Proof:* With notation as above, let

$$\langle a, b \rangle_* = \sum_{j=1}^k \langle P_j(a), P_j(b) \rangle_j + k \langle a, u \rangle \langle b, u \rangle.$$

This is clearly bilinear, invariant and symmetric. Indeed, since each  $\langle \cdot, \cdot \rangle_j$  is an inner product, so is  $\langle \cdot, \cdot \rangle_*$ . To see that it is positive on  $\mathbf{E}_+$ , note that for every  $x \in X \subseteq \mathbf{E}$ ,  $x = (\sum_j P_j x) + u/n$ , so, for  $x, y \in X$ , we have

$$\begin{aligned} \langle x, y \rangle_* &= \sum_j \langle x_j, y_j \rangle_j + k/n^2 \\ &= \sum_j \langle P_j(x) + u/n, P_j(y) + u/n \rangle_j \\ &= \sum_j \langle x_j, y_j \rangle_j \geq 0. \end{aligned}$$

Since  $X$  spans  $\mathbf{E}_+(A)$ ,  $\langle \cdot, \cdot \rangle_*$  is positive. The same computation shows that if  $x \perp y$ , so that  $x_j \perp y_j$  for each  $j$ , then, as  $\langle x_j, y_j \rangle_j = 0$  by hypothesis,  $\langle x, y \rangle_* = 0$ .  $\square$

*Proof of Theorem 4:* Let  $A = (X, \mathfrak{A}, \Omega, G)$  be a model in  $\mathcal{C}$ , and proceed as above to construct models  $A_j = (X_j, \mathfrak{A}_j, \Omega_j, G_j)$  corresponding to the irreducible components of  $u^\perp$  in  $\mathbf{E} = \mathbf{E}(A)$ . By Lemma 8,  $A_j$  is the image of  $A$  under a surjective homomorphism. Since  $\mathcal{C}$  is image-closed, each  $A_j$  lies in  $\mathcal{C}$ . Since  $\mathcal{C}$  is also dagger-monoidal, we have a canonical invariant bilinear form (4) on each  $\mathbf{E}(A)$ ,  $A \in \mathcal{C}$ , and this is orthogonalizing. Since  $A_j$  is irreducible, Corollary 2 tells us that this canonical form on  $\mathbf{E}(A_j) = \mathbf{E}_j$  must coincide with

standard form  $\langle, \rangle_j$  for all  $j = 1, \dots, n$ , whence, the SPIN inner product  $\langle, \rangle_*$  of Lemma 9 is orthogonalizing. Theorem 4 now follows from  $\square$

This gives us

**Corollary 5:** *If  $\mathcal{C}$  is an image-closed monoidal category of 2-symmetric models, admitting a  $\dagger$ -monoidal representation, then every state-closed, sharp model  $A \in \mathcal{C}$  is self-dual.*

## 6. Homogeneity

Let  $\mathcal{C}$  be an image-closed category of 2-symmetric probabilistic models. We've seen that if every model in  $\mathcal{C}$  has a conjugate, or if  $\mathcal{C}$  has a  $\dagger$ -monoidal representation, then for every state-closed, sharp model  $A \in \mathcal{C}$ , the cone  $\mathbf{E}(A)_+$  is self-dual. If this cone is also homogeneous, then the Koecher-Vinberg Theorem tells us that  $\mathbf{E}(A)_+$  is isomorphic to the cone of squares of a formally real Jordan algebra.

There are several ways in which to motivate the homogeneity of  $\mathbf{E}_+(A)$ , earlier explored in [30] and [7]. Before discussing these, let me mention one very direct interpretations of homogeneity. If we allow that all order-automorphisms  $\phi$  of  $\mathbf{E}^+ \simeq \mathbf{V}(\Omega)$  with  $u(\phi(\alpha)) \leq u(\alpha)$  represent legitimate physical processes, then homogeneity simply requires that it be possible to prepare any state in the interior of the cone, with non-zero probability, by applying a reversible physical process to the maximally mixed state. The main objection to simply taking this as a postulate is probably just that the use of the adjective "interior" here seems unaesthetic. (Then again, we seldom scruple to accord special axiomatic privileges to pure states.)

### 6.1 Self-Steering and Iso-Dilation

In [7] it is shown that homogeneity of the state cone follows from the assumption that every  $A \in \mathcal{C}$  is "self-steering":

**Definition 24 (Self-Steering):** A system  $A$  has the *Self-Steering property* iff every state  $\alpha \in \Omega(A)$  arises as the marginal of some bipartite state  $\omega \in \Omega(A \otimes A)$  that is *steering*, in the sense that, for every convex decomposition  $\sum_i t_i \alpha_i = \alpha$  of  $\alpha$  as the average of an ensemble of other states, there exists an observable  $\mathcal{E} = \{a_i\}$  on  $\mathbf{E}(A)$  with  $\omega(a_i, \cdot) = t_i \alpha_i$  for each  $i$ .

A less vivid, but mathematically simpler, assumption, also discussed in [7], is that every state in the interior of the state space, arise as the marginal of — or, in other language, can be *dilated to* — a bipartite *isomorphism state*, that is, a state  $\omega$  whose conditioning map,  $\widehat{\omega}$ , is an order-isomorphism  $\mathbf{E}^* \simeq \mathbf{E}$ . We might call this the *Iso-Dilation* condition. To see that this implies homogeneity, simply note that if  $\alpha$  and  $\beta$  are any two interior states (not necessarily normalized), then by assumption there exist bipartite states  $\omega_1$  and  $\omega_2$  with  $\widehat{\omega}_1(u) = \alpha$  and  $\widehat{\omega}_2(u) = \beta$ , whence,  $(\widehat{\omega}_2 \circ \widehat{\omega}_1^{-1})(\alpha) = \beta$ . Of course, there is still the (dubious?) aesthetic objection regarding the interior states.

That Self-Steering implies the homogeneity of the state cone is a consequence of the fact that any steering state on  $A \otimes A$  having a marginal lying in the interior of the state cone, must be an isomorphism state. *A priori*, then, Iso-Dilation is the weaker condition. When  $V(A)_+$  is irreducible, isomorphism states are pure, so this is a relative of the “purification postulate” of [11].

## 6.2 Full symmetry, correlation and filtering

Suppose  $A \in \mathcal{C}$  is sharp and *fully* symmetric, rather than only 2-symmetric. Then we can use the “correlation” and “filtering” axioms from [30] to secure the homogeneity of  $\mathbf{E}(A)_+$ . Recall that a bipartite state  $\omega$  on a composite  $AB$  *correlates* tests  $E \in \mathfrak{A}(A), F \in \mathfrak{A}(B)$  iff there is a bijection  $f : E \rightarrow F$  such that for all  $x, y \in E \times F$  with  $y \neq f(x)$ ,  $\omega(x, y) = 0$ . In other words, on  $E \times F$ ,  $\omega$  is supported on the graph of  $f$ . In this situation, I’ll say that  $\omega$  correlates  $E$  and  $F$  *along* the bijection  $f$ .

**Definition 25 (Correlation Condition):** A model  $A$  satisfies the *correlation condition* iff for every state  $\alpha$  on  $A$ , there exists a model  $B$ , a composite system  $AB$ , and a correlating bipartite state  $\omega$  on  $AB$  such that  $\omega_1 = \alpha$ .

The Correlation condition (a dilation principle, like Steering and Iso-dilation) is by no means obvious on purely operational grounds. On the other hand, something like it is needed if we are to be able to capture measurement processes “internally”, that is, in terms of the resources available in  $\mathcal{C}$ . For a further discussion of this point, see [29].

As noted in [29, 30], the correlation condition implies a kind of spectral decomposition for states:

**Lemma 10:** *Let  $A$  be sharp and satisfy correlation. Then for every state  $\alpha$  on  $\mathbf{E}$ , there exists a test  $E \in \mathfrak{A}(A)$  and convex coefficients  $t_x$  with  $\alpha = \sum_{x \in E} t_x \delta_x$ .*

*Proof:* Let  $\alpha = \omega_1$  where  $\omega$  correlates  $E$  with  $F$  along  $f$ . Then, by (4),

$$\alpha = \sum_{y \in F} \omega_2(y) \omega_{1|x} = \sum_{x \in E} \omega_2(f(x)) \delta_x.$$

Set  $\omega_2(f(x)) = t_x$ .  $\square$

**Definition 26 (Filtering Condition):**  $A$  satisfies the *filtering condition* iff for every test  $E \in \mathfrak{A}$ , and every set of constants  $0 < t_x \leq 1$ , there exists an affine automorphism  $\Phi \in \mathbf{E}_+$  with  $\Phi(x) = t_x x$ . I’ll call such an automorphism a *filter* on  $E$ .

Filtering is a reasonable assumption. If we think of a test  $E$  as, e.g., an array of detectors, then the axiom simply asserts that we can independently attenuate the reliabilities of these detectors — which, in practice, we can certainly do.

Now suppose that  $A$  is sharp, and let  $\delta_x$  denote the unique normalized state on  $\mathbf{E}$  with

$\delta_x(x) = 1$ . If  $\Phi$  is a filter on  $E \in \mathfrak{A}(A)$  with  $\Phi(x) = t_x x$ ,  $t_x > 0$ , then

$$\Phi^*(\delta_x)(x) = \delta_x(t_x x) = t_x \delta_x(x) = t_x,$$

and similarly,  $\Phi^*(\delta_x)(y) = 0$  for  $y \perp x$ . It follows that  $t_x^{-1} \Phi^*(\delta_x) = \delta_x$ , i.e.,  $\Phi^*(\delta_x) = t_x \delta_x$ .

As observed in [30], we now have

**Lemma 11:** *Let  $A$  be sharp, state-complete, fully  $G$ -symmetric, and satisfy both the correlation and filtering axioms. Then  $\mathbf{V}(A)_+$  is homogeneous.*

*Proof:* Let  $\alpha$  and  $\beta$  be normalized states in the interior of  $\mathbf{E}_+$ . We wish to find some order-automorphism of  $\mathbf{E}$  taking  $\alpha$  to  $\beta$ . By Lemma 10, we can expand  $\alpha$  and  $\beta$  as  $\alpha = \sum_{x \in E} t_x \delta_x$  and  $\beta = \sum_{y \in F} s_y \delta_y$  for some tests  $E, F \in \mathfrak{A}$ . Since  $\alpha$  and  $\beta$  are interior,  $t_x > 0$  and  $s_y > 0$  for all  $x \in E$  and  $y \in F$ . Let  $f : E \rightarrow F$  be any bijection, and let  $\Phi$  be a filter on  $E$  taking each  $x \in E$  to  $m_x x$ , where  $m_x = s_{f(x)}/t_x$ . Then we have

$$\Phi^*(\alpha) = \sum_{x \in E} t_x \Phi^*(\delta_x) = \sum_{x \in E} t_x m_x \delta_x = \sum_{x \in E} s_{f(x)} \delta_x.$$

By full symmetry,  $f$  extends to a symmetry  $g \in G$ ; applying this, we have

$$g\Phi^*(\alpha) = \sum_{x \in E} s_{f(x)} g\delta_x = \sum_{x \in E} s_{f(x)} \delta_{f(x)} = \sum_{y \in F} s_y \delta_y = \beta. \quad \square$$

Recall that if  $(\bar{A}, \gamma_A)$  is a *strong* conjugate for  $A$ , then for every state  $\alpha \in \Omega$ , there exists an equivariant state  $\omega^\alpha$  on  $A\bar{A}$  with  $\omega_1 = \alpha$ , and correlating some test  $E \in \mathfrak{A}(A)$  with the conjugate test  $\bar{E} \in \mathfrak{A}(\bar{A})$  along  $x \mapsto \bar{x} := \gamma_A(x)$ . This gives us the correlation property, and also, if  $A$  is sharp and state-complete, self-duality (by Theorem 1). Thus, we have

**Theorem 5:** *Let  $A$  be a sharp, state-complete, irreducible bi-symmetric model having a strong conjugate and satisfying the filtering condition. The  $\mathbf{E}(A)_+$  is homogeneous and self-dual.*

We also have

**Theorem 6:** *Let  $\mathcal{C}$  be an image-closed, dagger-monoidal probabilistic theory, in which every system is bi-symmetric and state-complete. If  $A \in \mathcal{C}$  is sharp and satisfies the Correlation and Filtering conditions, then  $\mathbf{E}(A)_+$  is homogeneous and self-dual.*

(Notice that here, as in Theorem 4b, image-closure can be dropped, if we are willing to concentrate on incompressible models.)



## 6 Conclusion and Speculations

The foregoing results show that the Jordan structure of finite-dimensional QM emerges very naturally from a few relatively simple constraints having reasonably clear operational or physical meanings. Or, better to say, follow from *any of several different* clusters, or packages, of such constraints. Two of these are given in Theorems 5 and 6. Some others:

- (1) Individual systems are bi-symmetric, state-closed, irreducible, and has a conjugate system with an iso-correlator. Every interior state can be reversibly prepared from the maximally mixed state.
- (2) Individual systems are sharp, state-closed, irreducible, fully symmetric, and satisfy both the strong correlation and the filtering condition.
- (3) Systems collectively form an image-closed category with a dagger-monoidal representation, and individually are sharp, bi-symmetric and satisfy the steering condition.

Obviously, though, there's much left to do. Regarding (3), for example, while existence of a symmetric monoidal structure is not usually viewed as problematic, the existence of a dagger cries out for further explanation. One would like to find a compelling physical or operational interpretation for such a structure. (One attractive, though at this point vague, idea is that a dagger corresponds to a global time-reversal symmetry.)

To all of these examples, there is an aesthetic objection: there are too many moving parts. It is likely, however, that the apparatus can be simplified. For example, there is a sense in which both full symmetry and filtering are expressions of the same idea: that any classically allowed, reversible process acting on the probabilistic apparatus associated with a single test, should extend to an irreversible process acting on the entire system. In terms of a slogan: any classically reversible process corresponds to a physically reversible process. Finally, it would be very desirable to replace the image-closure condition with some kind of reduction theory, according to which all systems in  $\mathcal{C}$  simply *are* direct sums, in some suitable sense, of irreducible systems. At present, I do not see how to obtain such a theory by anything short of *fiat*, but this may simply reflect lack of sufficient effort, or wit, on my part.<sup>16</sup>

I have made no real effort to establish in detail how the various conditions enumerated here depend on one another, so there is the possibility that, given some of them, others are

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<sup>16</sup>Alternatively, one could hope to show that (perhaps in the presence of other constraints), homogeneity already implies irreducibility. This is true, for example, if the group  $G$  comprises all unit-preserving order-automorphisms in the connected component of the identity of  $\text{Aut}(\mathbf{E})$ .

simply redundant. It is also perfectly conceivable that these conditions are stronger than necessary. For example, I haven't checked to see whether every simple Jordan model has a conjugate, or satisfies filtering. At a more fundamental level, it remains an important open question whether there exist any *non- $C^*$* -algebraic dagger-symmetric monoidal categories of formally real Jordan algebras.

I want to emphasize again that local tomography has played no role here. In a forthcoming paper [9] with Howard Barnum, it will be shown that if  $\mathcal{E}$  is a dagger-HSD category of order-unit spaces with non-signaling, locally tomographic composites, and if  $\mathcal{E}$  contains a model having the structure of a qubit, then it is a category of finite-dimensional complex matrix algebras.

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